Fourth Edition

LINEAR ALGEBRA AND ITS APPLICATIONS

Gilbert Strang

THOMSON
BROOKS/COLE

Linear Algebra and Its Applications, Fourth Edition
Gilbert Strang

Acquisitions Editor: *John-Paul Ramin*
Assistant Editor: *Katherine Brayton*
Editorial Assistant: *Leata Holloway*
Marketing Manager: *Tom Ziolkowski*
Marketing Assistant: *Jennifer Velasquez*
Marketing Communications Manager: *Bryan Vann*
Senior Project Manager, Editorial Production: *Janet Hill*
Senior Art Director: *Vernon Boes*
Print Buyer: *Lisa Claudeanos*
Permissions Editor: *Audrey Pettengill*
Production Editor: *Rozi Harris, ICC*
Text Designer: *Kim Rokusek*
Illustrator: *Brett Coonley/ICC*
Cover Designer: *Laurie Albrecht*
Cover Image: *Judith Laurel Harkness*
Compositor: *Interactive Composition Corporation*
Cover/Interior Printer: *R.R. Donnelley/Crawfordsville*

Printed in the United States of America

2 3 4 5 6 7 09 08 07 06 05

Library of Congress Control Number: 2005923623

Student Edition: ISBN 0-03-010567-6

Thomson Higher Education
10 Davis Drive
Belmont, CA 94002-3098
USA

Asia (including India)
Thomson Learning
5 Shenton Way
#01-01 UIC Building
Singapore 068808

Australia/New Zealand
Thomson Learning Australia
102 Dodds Street
Southbank, Victoria 3006
Australia

Canada
Thomson Nelson
1120 Birchmount Road
Toronto, Ontario M1K 5G4
Canada

UK/Europe/Middle East/Africa
Thomson Learning
High Holborn House
50/51 Bedford Row
London WC1R 4LR
United Kingdom

Latin America
Thomson Learning
Seneca, 53
Colonia Polanco
11560 Mexico
D.F. Mexico

Spain (including Portugal)
Thomson Paraninfo
Calle Magallanes, 25
28015 Madrid, Spain

Table of Contents

Preface

Revising this textbook has been a special challenge, for a very nice reason. So many people have read this book, and taught from it, and even loved it. The spirit of the book could never change. This text was written to help our teaching of linear algebra keep up with the enormous importance of this subject—which just continues to grow.

One step was certainly possible and desirable—*to add new problems*. Teaching for all these years required hundreds of new exam questions (especially with quizzes going onto the web). I think you will approve of the extended choice of problems. The questions are still a mixture of *explain* and *compute*—the two complementary approaches to learning this beautiful subject.

I personally believe that many more people need linear algebra than calculus. Isaac Newton might not agree! But he isn't teaching mathematics in the 21st century (and maybe he wasn't a great teacher, but we will give him the benefit of the doubt). Certainly the laws of physics are well expressed by differential equations. Newton needed calculus—quite right. But the scope of science and engineering and management (and life) is now so much wider, and linear algebra has moved into a central place.

May I say a little more, because many universities have not yet adjusted the balance toward linear algebra. Working with curved lines and curved surfaces, the first step is always to *linearize*. Replace the curve by its tangent line, fit the surface by a plane, and the problem becomes linear. The power of this subject comes when you have ten variables, or 1000 variables, instead of two.

You might think I am exaggerating to use the word "beautiful" for a basic course in mathematics. Not at all. This subject begins with two vectors v and w, pointing in different directions. The key step is to *take their linear combinations*. We multiply to get $3v$ and $4w$, and we add to get the particular combination $3v + 4w$. That new vector is in the *same plane* as v and w. When we take all combinations, we are filling in the whole plane. If I draw v and w on this page, their combinations $cv + dw$ fill the page (and beyond), but they *don't go up* from the page.

In the language of linear equations, I can solve $cv + dw = b$ exactly when the vector b lies in the same plane as v and w.

Matrices

I will keep going a little more to convert combinations of three-dimensional vectors into linear algebra. If the vectors are $v = (1, 2, 3)$ and $w = (1, 3, 4)$, put them into the **columns of a matrix**:

$$\textbf{matrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}.$$

To find combinations of those columns, "**multiply**" the matrix by a vector (c, d):

$$\textbf{Linear combinations } \boldsymbol{cv + dw} \qquad \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Those combinations fill a *vector space*. We call it the **column space** of the matrix. (For these two columns, that space is a plane.) To decide if $b = (2, 5, 7)$ is on that plane, we have three components to get right. So we have three equations to solve:

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \quad \text{means} \quad \begin{aligned} c + d &= 2 \\ 2c + 3d &= 5 \\ 3c + 4d &= 7. \end{aligned}$$

I leave the solution to you. The vector $b = (2, 5, 7)$ does lie in the plane of v and w. If the 7 changes to any other number, then b won't lie in the plane—it will *not* be a combination of v and w, and the three equations will have no solution.

Now I can describe the first part of the book, about linear equations $Ax = b$. The matrix A has n columns and m rows. *Linear algebra moves steadily to n vectors in m-dimensional space.* We still want combinations of the columns (in the column space). We still get m equations to produce b (one for each row). Those equations may or may not have a solution. They always have a least-squares solution.

The interplay of columns and rows is the heart of linear algebra. It's not totally easy, but it's not too hard. Here are four of the central ideas:

1. The **column space** (all combinations of the columns).
2. The **row space** (all combinations of the rows).
3. The **rank** (the number of independent columns) (or rows).
4. **Elimination** (the good way to find the rank of a matrix).

I will stop here, so you can start the course.

Web Pages

It may be helpful to mention the web pages connected to this book. So many messages come back with suggestions and encouragement, and I hope you will make free use of everything. You can directly access *http://web.mit.edu/18.06*, which is continually updated for the course that is taught every semester. Linear algebra is also on MIT's OpenCourseWare site *http://ocw.mit.edu*, where 18.06 became exceptional by including videos of the lectures (which you definitely don't have to watch...). Here is a part of what is available on the web:

1. Lecture schedule and current homeworks and exams with solutions.
2. The goals of the course, and conceptual questions.
3. Interactive Java demos (audio is now included for eigenvalues).
4. Linear Algebra Teaching Codes and MATLAB problems.
5. Videos of the complete course (taught in a real classroom).

The course page has become a valuable link to the class, and a resource for the students. I am very optimistic about the potential for graphics with sound. The bandwidth for voiceover is low, and FlashPlayer is freely available. This offers a *quick review* (with active experiment), and the full lectures can be downloaded. I hope professors and students worldwide will find these web pages helpful. My goal is to make this book as useful as possible with all the course material I can provide.

Other Supporting Materials

Student Solutions Manual 0-495-01325-0 The Student Solutions Manual provides solutions to the odd-numbered problems in the text.

Instructor's Solutions Manual 0-030-10568-4 The Instructor's Solutions Manual has teaching notes for each chapter and solutions to all of the problems in the text.

Structure of the Course

The two fundamental problems are $Ax = b$ and $Ax = \lambda x$ for square matrices A. The first problem $Ax = b$ has a solution when A has *independent columns*. The second problem $Ax = \lambda x$ looks for *independent eigenvectors*. A crucial part of this course is to learn what "independence" means.

I believe that most of us learn first from examples. You can see that

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \qquad \textbf{does } \textbf{\textit{not}} \textbf{ have independent columns}.$$

Column 1 plus column 2 equals column 3. A wonderful theorem of linear algebra says that the three rows are not independent either. The third row must lie in the same plane as the first two rows. Some combination of rows 1 and 2 will produce row 3. You might find that combination quickly (I didn't). In the end I had to use elimination to discover that the right combination uses 2 times row 2, minus row 1.

Elimination is the simple and natural way to understand a matrix by producing a lot of zero entries. So the course starts there. But don't stay there too long! You have to get from combinations of the rows, to independence of the rows, to "dimension of the row space." That is a key goal, to see whole spaces of vectors: the ***row space*** and the ***column space*** and the ***nullspace.***

A further goal is to understand how the matrix *acts*. When A multiplies x it produces the new vector Ax. The whole space of vectors moves—it is "transformed" by A. Special transformations come from particular matrices, and those are the foundation stones of linear algebra: diagonal matrices, orthogonal matrices, triangular matrices, symmetric matrices.

The eigenvalues of those matrices are special too. I think 2 by 2 matrices provide terrific examples of the information that eigenvalues λ can give. Sections 5.1 and 5.2 are worth careful reading, to see how $Ax = \lambda x$ is useful. Here is a case in which small matrices allow tremendous insight.

Overall, the beauty of linear algebra is seen in so many different ways:

1. **Visualization.** Combinations of vectors. Spaces of vectors. Rotation and reflection and projection of vectors. Perpendicular vectors. Four fundamental subspaces.

2. **Abstraction.** Independence of vectors. Basis and dimension of a vector space. Linear transformations. Singular value decomposition and the best basis.

3. **Computation.** Elimination to produce zero entries. Gram–Schmidt to produce orthogonal vectors. Eigenvalues to solve differential and difference equations.

4. **Applications.** Least-squares solution when $Ax = b$ has too many equations. Difference equations approximating differential equations. Markov probability matrices (the basis for Google!). Orthogonal eigenvectors as principal axes (and more...).

To go further with those applications, may I mention the books published by Wellesley-Cambridge Press. They are all linear algebra in disguise, applied to signal processing and partial differential equations and scientific computing (and even GPS). If you look at *http://www.wellesleycambridge.com*, you will see part of the reason that linear algebra is so widely used.

After this preface, the book will speak for itself. You will see the spirit right away. The emphasis is on understanding—*I try to explain rather than to deduce*. This is a book about real mathematics, not endless drill. In class, I am constantly working with examples to teach what students need.

Acknowledgments

I enjoyed writing this book, and I certainly hope you enjoy reading it. A big part of the pleasure comes from working with friends. I had wonderful help from Brett Coonley and Cordula Robinson and Erin Maneri. They created the LaTeX files and drew all the figures. Without Brett's steady support I would never have completed this new edition.

Earlier help with the Teaching Codes came from Steven Lee and Cleve Moler. Those follow the steps described in the book; MATLAB and Maple and Mathematica are faster for large matrices. All can be used (*optionally*) in this course. I could have added "Factorization" to that list above, as a fifth avenue to the understanding of matrices:

[L,U,P] = lu(A)	for linear equations
[Q,R] = qr(A)	to make the columns orthogonal
[S,E] = eig(A)	to find eigenvectors and eigenvalues.

In giving thanks, I never forget the first dedication of this textbook, years ago. That was a special chance to thank my parents for so many unselfish gifts. Their example is an inspiration for my life.

And I thank the reader too, hoping you like this book.

Gilbert Strang

Chapter

1 Matrices and Gaussian Elimination

1.1 INTRODUCTION

This book begins with the central problem of linear algebra: *solving linear equations*. The most important case, and the simplest, is when the number of unknowns equals the number of equations. We have ***n* equations in *n* unknowns**, starting with $n = 2$:

$$\begin{array}{llcrcr} \textbf{Two equations} & 1x & + & 2y & = & 3 \\ \textbf{Two unknowns} & 4x & + & 5y & = & 6. \end{array} \tag{1}$$

The unknowns are x and y. I want to describe two ways, *elimination* and *determinants*, to solve these equations. Certainly x and y are determined by the numbers $1, 2, 3, 4, 5, 6$. The question is how to use those six numbers to solve the system.

1. Elimination Subtract 4 times the first equation from the second equation. This eliminates x from the second equation, and it leaves one equation for y:

$$\textbf{(equation 2)} - \textbf{4(equation 1)} \qquad -3y = -6. \tag{2}$$

Immediately we know $y = 2$. Then x comes from the first equation $1x + 2y = 3$:

$$\textbf{Back-substitution} \qquad 1x + 2(2) = 3 \qquad \text{gives} \qquad x = -1. \tag{3}$$

Proceeding carefully, we check that x and y also solve the second equation. This should work and it does: 4 times $(x = -1)$ plus 5 times $(y = 2)$ equals 6.

2. Determinants The solution $y = 2$ depends completely on those six numbers in the equations. There must be a formula for y (and also x). It is a "ratio of determinants" and I hope you will allow me to write it down directly:

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{1 \cdot 6 - 3 \cdot 4}{1 \cdot 5 - 2 \cdot 4} = \frac{-6}{-3} = 2. \tag{4}$$

That could seem a little mysterious, unless you already know about 2 by 2 determinants. They gave the same answer $y = 2$, coming from the same ratio of -6 to -3. If we stay with determinants (which we don't plan to do), there will be a similar formula to compute the other unknown, x:

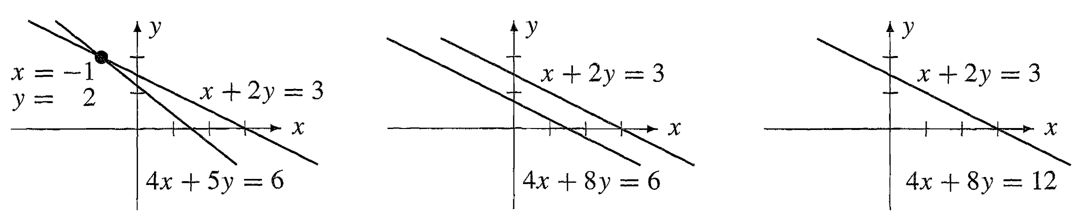
$$x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{3 \cdot 5 - 2 \cdot 6}{1 \cdot 5 - 2 \cdot 4} = \frac{3}{-3} = -1. \tag{5}$$

Let me compare those two approaches, looking ahead to real problems when n is much larger ($n = 1000$ is a very moderate size in scientific computing). The truth is that direct use of the determinant formula for 1000 equations would be a total disaster. It would use the million numbers on the left sides correctly, but not efficiently. We will find that formula (Cramer's Rule) in Chapter 4, but we want a good method to solve 1000 equations in Chapter 1.

That good method is ***Gaussian Elimination***. This is the algorithm that is constantly used to solve large systems of equations. From the examples in a textbook ($n = 3$ is close to the upper limit on the patience of the author and reader) you might not see much difference. Equations (2) and (4) used essentially the same steps to find $y = 2$. Certainly x came faster by the back-substitution in equation (3) than the ratio in (5). For larger n there is absolutely no question. Elimination wins (and this is even the best way to compute determinants).

The idea of elimination is deceptively simple—you will master it after a few examples. It will become the basis for half of this book, simplifying a matrix so that we can understand it. Together with the mechanics of the algorithm, we want to explain four deeper aspects in this chapter. They are:

1. Linear equations lead to ***geometry of planes***. It is not easy to visualize a nine-dimensional plane in ten-dimensional space. It is harder to see ten of those planes, intersecting at the solution to ten equations—but somehow this is almost possible. Our example has two lines in Figure 1.1, meeting at the point $(x, y) = (-1, 2)$. Linear algebra moves that picture into ten dimensions, where the intuition has to imagine the geometry (and gets it right).
2. We move to ***matrix notation***, writing the n unknowns as a vector x and the n equations as $Ax = b$. We multiply A by "elimination matrices" to reach an upper triangular matrix U. Those steps factor A into L *times* U, where L is lower



One solution $(x, y) = (-1, 2)$ **Parallel: No solution** **Whole line of solutions**

Figure 1.1 The example has one solution. Singular cases have none or too many.

triangular. I will write down A and its factors for our example, and explain them at the right time:

$$\textbf{Factorization} \qquad A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = \textbf{\textit{L times U}}. \qquad (6)$$

First we have to introduce matrices and vectors and the rules for multiplication. Every matrix has a ***transpose*** $\boldsymbol{A}^{\mathrm{T}}$. This matrix has an ***inverse*** $\boldsymbol{A}^{-1}$.

3. In most cases elimination goes forward without difficulties. The matrix has an inverse and the system $Ax = b$ has one solution. In exceptional cases the method will *break down*—either the equations were written in the wrong order, which is easily fixed by exchanging them, or the equations don't have a unique solution.

   That ***singular case*** will appear if 8 replaces 5 in our example:

$$\begin{array}{lrcrcl} \textbf{Singular case} & 1x & + & 2y & = & 3 \\ \textbf{Two parallel lines} & 4x & + & \mathbf{8}y & = & 6. \end{array} \qquad (7)$$

   Elimination still innocently subtracts 4 times the first equation from the second. But look at the result!

$$\textbf{(equation 2)} - \textbf{4(equation 1)} \qquad 0 = -6.$$

   This singular case has ***no solution***. Other singular cases have ***infinitely many solutions***. (Change 6 to 12 in the example, and elimination will lead to $0 = 0$. Now y can have *any value*.) When elimination breaks down, we want to find every possible solution.

4. We need a rough count of the ***number of elimination steps*** required to solve a system of size n. The computing cost often determines the accuracy in the model. A hundred equations require a third of a million steps (multiplications and subtractions). The computer can do those quickly, but not many trillions. And already after a million steps, roundoff error could be significant. (Some problems are sensitive; others are not.) Without trying for full detail, we want to see large systems that arise in practice, and how they are actually solved.

The final result of this chapter will be an elimination algorithm that is about as efficient as possible. It is essentially the algorithm that is in constant use in a tremendous variety of applications. And at the same time, understanding it in terms of *matrices*—the coefficient matrix A, the matrices E for elimination and P for row exchanges, and the final factors L and U—is an essential foundation for the theory. I hope you will enjoy this book and this course.

1.2 THE GEOMETRY OF LINEAR EQUATIONS

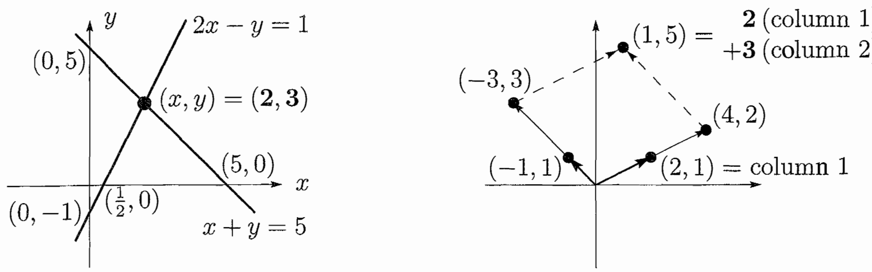
The way to understand this subject is by example. We begin with two extremely humble equations, recognizing that you could solve them without a course in linear algebra. Nevertheless I hope you will give Gauss a chance:

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5. \end{aligned}$$

We can look at that system *by rows or by columns*. We want to see them both.

The first approach concentrates on the separate equations (the ***rows***). That is the most familiar, and in two dimensions we can do it quickly. The equation $2x - y = 1$ is represented by a *straight line* in the x-y plane. The line goes through the points $x = 1$, $y = 1$ and $x = \frac{1}{2}$, $y = 0$ (and also through (2, 3) and all intermediate points). The second equation $x + y = 5$ produces a second line (Figure 1.2a). Its slope is $dy/dx = -1$ and it crosses the first line at the solution.

The point of intersection lies on both lines. It is the only solution to both equations. That point $x = 2$ and $y = 3$ will soon be found by "elimination."



(a) **Lines meet at $x = 2, y = 3$** (b) **Columns combine with 2 and 3**

Figure 1.2 Row picture (two lines) and column picture (combine columns).

The second approach looks at the ***columns*** of the linear system. The two separate equations are really ***one vector equation***:

$$\textbf{Column form} \qquad x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}.$$

The problem is ***to find the combination of the column vectors on the left side that produces the vector on the right side.*** Those vectors (2, 1) and (−1, 1) are represented by the bold lines in Figure 1.2b. The unknowns are the numbers x and y that multiply the column vectors. The whole idea can be seen in that figure, where 2 times column 1 is added to 3 times column 2. Geometrically this produces a famous parallelogram. Algebraically it produces the correct vector (1, 5), on the right side of our equations. The column picture confirms that $x = 2$ and $y = 3$.

More time could be spent on that example, but I would rather move forward to $n = 3$. Three equations are still manageable, and they have much more variety:

$$\textbf{Three planes} \qquad \begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \phantom{{}+ 2w} &= -2 \\ -2u + 7v + 2w &= 9. \end{aligned} \tag{1}$$

Again we can study the rows or the columns, and we start with the rows. Each equation describes a ***plane*** in three dimensions. The first plane is $2u + v + w = 5$, and it is sketched in Figure 1.3. It contains the points $(\frac{5}{2}, 0, 0)$ and (0, 5, 0) and (0, 0, 5). It is determined by any three of its points—provided they do not lie on a line.

Changing 5 *to* 10, *the plane* $2u + v + w = 10$ *would be parallel to this one.* It contains (5, 0, 0) and (0, 10, 0) and (0, 0, 10), twice as far from the origin—which is

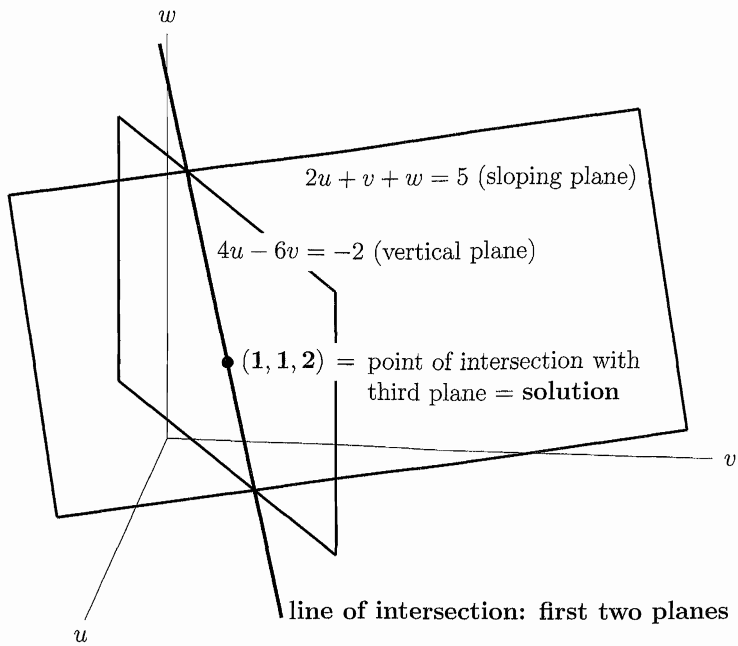


Figure 1.3 The row picture: three intersecting planes from three linear equations.

the center point $u = 0$, $v = 0$, $w = 0$. Changing the right side moves the plane parallel to itself, and the plane $2u + v + w = 0$ goes through the origin.

The second plane is $4u - 6v = -2$. It is drawn vertically, because w can take any value. The coefficient of w is zero, but this remains a plane in 3-space. (The equation $4u = 3$, or even the extreme case $u = 0$, would still describe a plane.) The figure shows the intersection of the second plane with the first. That intersection is a line. *In three dimensions a line requires two equations*; in n dimensions it will require $n - 1$.

Finally the third plane intersects this line in a point. The plane (not drawn) represents the third equation $-2u + 7v + 2w = 9$, and it crosses the line at $u = 1$, $v = 1$, $w = 2$. That triple intersection point $(1, 1, 2)$ solves the linear system.

How does this row picture extend into n dimensions? The n equations will contain n unknowns. The first equation still determines a "plane." It is no longer a two-dimensional plane in 3-space; somehow it has "dimension" $n - 1$. It must be flat and extremely thin within n-dimensional space, although it would look solid to us.

If time is the fourth dimension, then the plane $t = 0$ cuts through four-dimensional space and produces the three-dimensional universe we live in (or rather, the universe as it was at $t = 0$). Another plane is $z = 0$, which is also three-dimensional; it is the ordinary x-y plane taken over all time. Those three-dimensional planes will intersect! They share the ordinary x-y plane at $t = 0$. We are down to two dimensions, and the next plane leaves a line. Finally a fourth plane leaves a single point. It is the intersection point of 4 planes in 4 dimensions, and it solves the 4 underlying equations.

I will be in trouble if that example from relativity goes any further. The point is that linear algebra can operate with any number of equations. The first equation produces an $(n - 1)$-dimensional plane in n dimensions. The second plane intersects it (we hope) in a smaller set of "dimension $n-2$." Assuming all goes well, every new plane (every new equation) reduces the dimension by one. At the end, when all n planes are accounted for, the intersection has dimension zero. It is a *point*, it lies on all the planes, and its coordinates satisfy all n equations. It is the solution!

Column Vectors and Linear Combinations

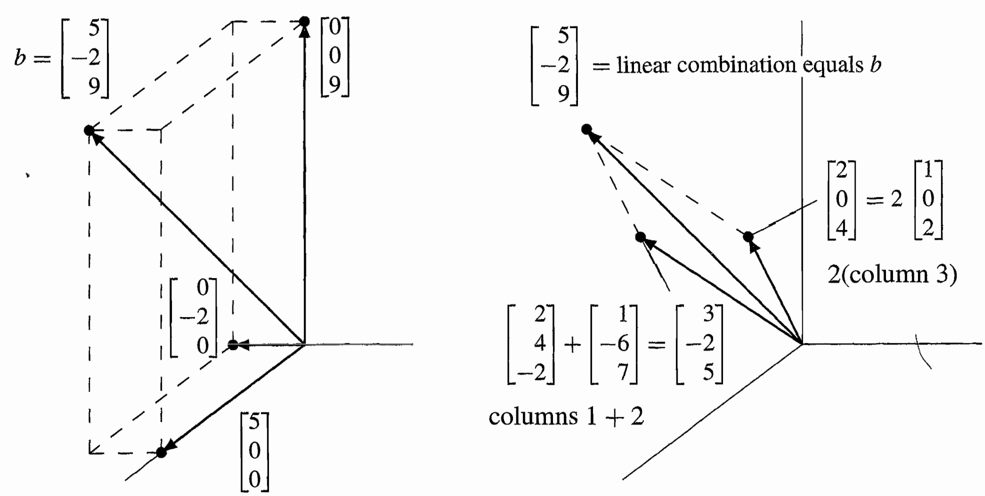
We turn to the columns. This time the vector equation (the same equation as (1)) is

$$\textbf{Column form} \qquad u\begin{bmatrix}2\\4\\-2\end{bmatrix}+v\begin{bmatrix}1\\-6\\7\end{bmatrix}+w\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}=b. \tag{2}$$

Those are *three-dimensional column vectors. **The vector b is identified with the point whose coordinates are*** **5, −2, 9**. Every point in three-dimensional space is matched to a vector, and vice versa. That was the idea of Descartes, who turned geometry into algebra by working with the coordinates of the point. We can write the vector in a column, or we can list its components as $b=(5,-2,9)$, or we can represent it geometrically by an arrow from the origin. You can choose *the arrow*, or *the point*, or *the three numbers*. In six dimensions it is probably easiest to choose the six numbers.

We use parentheses and commas when the components are listed horizontally, and square brackets (with no commas) when a column vector is printed vertically. What really matters is ***addition of vectors*** and ***multiplication by a scalar*** (a number). In Figure 1.4a you see a vector addition, component by component:

$$\textbf{Vector addition} \qquad \begin{bmatrix}5\\0\\0\end{bmatrix}+\begin{bmatrix}0\\-2\\0\end{bmatrix}+\begin{bmatrix}0\\0\\9\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}.$$



(a) **Add vectors along axes** (b) **Add columns 1 + 2 + (3 + 3)**

Figure 1.4 The column picture: linear combination of columns equals b.

In the right-hand figure there is a multiplication by 2 (and if it had been -2 the vector would have gone in the reverse direction):

$$\textbf{Multiplication by scalars}\qquad 2\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}2\\0\\4\end{bmatrix},\qquad -2\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}-2\\0\\-4\end{bmatrix}.$$

Also in the right-hand figure is one of the central ideas of linear algebra. It uses *both* of the basic operations; vectors are *multiplied by numbers and then added.* The result is called a ***linear combination***, and this combination solves our equation:

$$\textbf{Linear combination}\qquad \mathbf{1}\begin{bmatrix}2\\4\\-2\end{bmatrix}+\mathbf{1}\begin{bmatrix}1\\-6\\7\end{bmatrix}+\mathbf{2}\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}.$$

Equation (2) asked for multipliers u, v, w that produce the right side b. Those numbers are $u = \mathbf{1}$, $v = \mathbf{1}$, $w = \mathbf{2}$. They give the correct combination of the columns. They also gave the point $(1, 1, 2)$ in the row picture (where the three planes intersect).

Our true goal is to look beyond two or three dimensions into n dimensions. With n equations in n unknowns, there are n planes in the row picture. There are n vectors in the column picture, plus a vector b on the right side. The equations ask for a ***linear combination of the n columns that equals b***. For certain equations that will be impossible. Paradoxically, the way to understand the good case is to study the bad one. Therefore we look at the geometry exactly when it breaks down, in the **singular case**.

Row picture: Intersection of planes	***Column picture***: Combination of columns

The Singular Case

Suppose we are again in three dimensions, and the three planes in the row picture *do not intersect*. What can go wrong? One possibility is that two planes may be parallel. The equations $2u + v + w = 5$ and $4u + 2v + 2w = 11$ are inconsistent—and parallel planes give no solution (Figure 1.5a shows an end view). In two dimensions, parallel lines are the only possibility for breakdown. But three planes in three dimensions can be in trouble without being parallel.

The most common difficulty is shown in Figure 1.5b. From the end view the planes form a triangle. Every pair of planes intersects in a line, and those lines are parallel. The

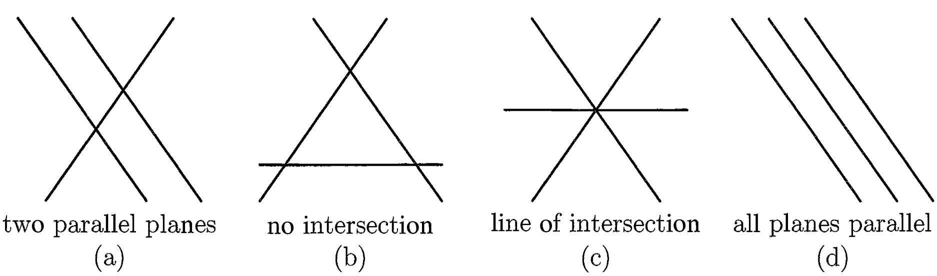


Figure 1.5 Singular cases: no solution for (a), (b), or (d), an infinity of solutions for (c).

third plane is not parallel to the other planes, but it is parallel to their line of intersection. This corresponds to a singular system with $b = (2, 5, 6)$:

No solution, as in Figure 1.5b
$$\begin{aligned} u + v + w &= 2 \\ 2u + 3w &= 5 \\ 3u + v + 4w &= 6. \end{aligned} \tag{3}$$

The first two left sides add up to the third. On the right side that fails: $2 + 5 \neq 6$. Equation 1 plus equation 2 minus equation 3 is the impossible statement $0 = 1$. Thus the equations are ***inconsistent***, as Gaussian elimination will systematically discover.

Another singular system, close to this one, has an **infinity of solutions**. When the 6 in the last equation becomes 7, the three equations combine to give $0 = 0$. Now the third equation is the sum of the first two. In that case the three planes have a whole *line in common* (Figure 1.5c). Changing the right sides will move the planes in Figure 1.5b parallel to themselves, and for $b = (2, 5, 7)$ the figure is suddenly different. The lowest plane moved up to meet the others, and there is a line of solutions. Problem 1.5c is still singular, but now it suffers from ***too many solutions*** instead of too few.

The extreme case is three parallel planes. For most right sides there is no solution (Figure 1.5d). For special right sides (like $b = (0, 0, 0)$!) there is a whole plane of solutions—because the three parallel planes move over to become the same.

What happens to the ***column picture*** when the system is singular? It has to go wrong; the question is how. There are still three columns on the left side of the equations, and we try to combine them to produce b. Stay with equation (3):

Singular case: Column picture
Three columns in the same plane
Solvable only for b in that plane
$$u \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + v \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b. \tag{4}$$

For $b = (2, 5, 7)$ this was possible; for $b = (2, 5, 6)$ it was not. The reason is that ***those three columns lie in a plane***. Then every combination is also in the plane (which goes through the origin). If the vector b is not in that plane, no solution is possible (Figure 1.6). That is by far the most likely event; a singular system generally has no solution. But

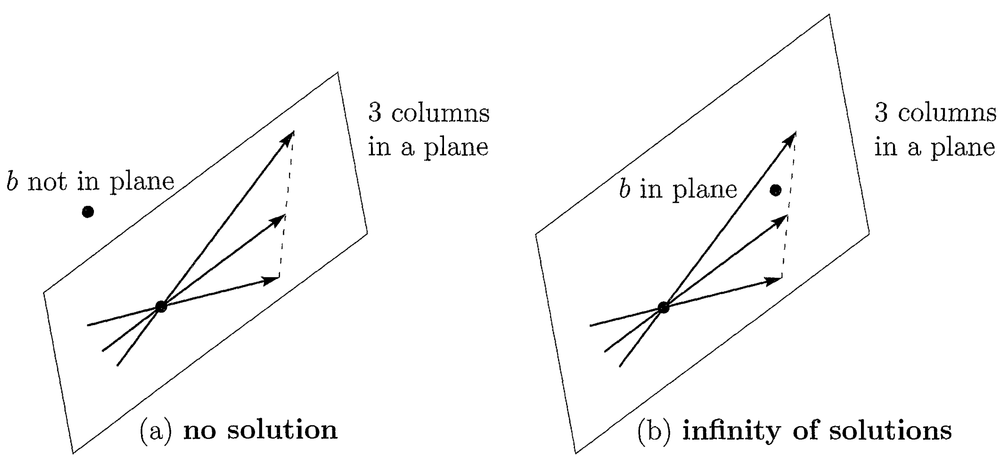


Figure 1.6 Singular cases: b outside or inside the plane with all three columns.

there is a chance that b *does* lie in the plane of the columns. In that case there are too many solutions; the three columns can be combined in *infinitely many ways* to produce b. That column picture in Figure 1.6b corresponds to the row picture in Figure 1.5c.

How do we know that the three columns lie in the same plane? One answer is to find a combination of the columns that adds to zero. After some calculation, it is $u = 3$, $v = -1$, $w = -2$. Three times column 1 equals column 2 plus twice column 3. Column 1 is in the plane of columns 2 and 3. Only two columns are independent.

The vector $b = (2, 5, 7)$ is in that plane of the columns—it is column 1 plus column 3—so $(1, 0, 1)$ is a solution. *We can add any multiple of the combination* $(3, -1, -2)$ *that gives* $b = 0$. So there is a whole line of solutions—as we know from the row picture.

The truth is that we *knew* the columns would combine to give zero, because the rows did. That is a fact of mathematics, not of computation—and it remains true in dimension n. ***If the n planes have no point in common, or infinitely many points, then the n columns lie in the same plane.***

If the row picture breaks down, so does the column picture. That brings out the difference between Chapter 1 and Chapter 2. This chapter studies the most important problem—the *nonsingular* case—where there is one solution and it has to be found. Chapter 2 studies the general case, where there may be many solutions or none. In both cases we cannot continue without a decent notation (*matrix notation*) and a decent algorithm (*elimination*). After the exercises, we start with elimination.

Problem Set 1.2

1. For the equations $x + y = 4$, $2x - 2y = 4$, draw the row picture (two intersecting lines) and the column picture (combination of two columns equal to the column vector $(4, 4)$ on the right side).

2. Solve to find a combination of the columns that equals b:

Triangular system
$$\begin{aligned} u - v - w &= b_1 \\ v + w &= b_2 \\ w &= b_3. \end{aligned}$$

3. (Recommended) Describe the intersection of the three planes $u + v + w + z = 6$ and $u + w + z = 4$ and $u + w = 2$ (all in four-dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane $u = -1$ is included? Find a fourth equation that leaves us with no solution.

4. Sketch these three lines and decide if the equations are solvable:

3 by 2 system
$$\begin{aligned} x + 2y &= 2 \\ x - y &= 2 \\ y &= 1. \end{aligned}$$

What happens if all right-hand sides are zero? Is there any nonzero choice of right-hand sides that allows the three lines to intersect at the same point?

5. Find two points on the line of intersection of the three planes $t = 0$ and $z = 0$ and $x + y + z + t = 1$ in four-dimensional space.

6. When $b = (2, 5, 7)$, find a solution (u, v, w) to equation (4) different from the solution $(1, 0, 1)$ mentioned in the text.

7. Give two more right-hand sides in addition to $b = (2, 5, 7)$ for which equation (4) can be solved. Give two more right-hand sides in addition to $b = (2, 5, 6)$ for which it cannot be solved.

8. Explain why the system

$$\begin{aligned} u + v + w &= 2 \\ u + 2v + 3w &= 1 \\ v + 2w &= 0 \end{aligned}$$

is singular by finding a combination of the three equations that adds up to $0 = 1$. What value should replace the last zero on the right side to allow the equations to have solutions—and what is one of the solutions?

9. The column picture for the previous exercise (singular system) is

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = b.$$

Show that the three columns on the left lie in the same plane by expressing the third column as a combination of the first two. What are all the solutions (u, v, w) if b is the zero vector $(0, 0, 0)$?

10. (Recommended) Under what condition on y_1, y_2, y_3 do the points $(0, y_1)$, $(1, y_2)$, $(2, y_3)$ lie on a straight line?

11. These equations are certain to have the solution $x = y = 0$. For which values of a is there a whole line of solutions?

$$\begin{aligned} ax + 2y &= 0 \\ 2x + ay &= 0 \end{aligned}$$

12. Starting with $x + 4y = 7$, find the equation for the parallel line through $x = 0$, $y = 0$. Find the equation of another line that meets the first at $x = 3$, $y = 1$.

Problems 13–15 are a review of the row and column pictures.

13. Draw the two pictures in two planes for the equations $x - 2y = 0$, $x + y = 6$.

14. For two linear equations in three unknowns x, y, z, the row picture will show (2 or 3) (lines or planes) in (two or three)-dimensional space. The column picture is in (two or three)-dimensional space. The solutions normally lie on a ______.

15. For four linear equations in two unknowns x and y, the row picture shows four ______. The column picture is in ______-dimensional space. The equations have no solution unless the vector on the right-hand side is a combination of ______.

16. Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$ and a third point halfway between.

17. The first of these equations plus the second equals the third:

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 5. \end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also ______. The equations have infinitely many solutions (the whole line **L**). Find three solutions.

18. Move the third plane in Problem 17 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not*? The first two planes meet along the line **L**, but the third plane doesn't ______ that line.

19. In Problem 17 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a "singular case" because the third column is ______. Find two combinations of the columns that give $b = (2, 3, 5)$. This is only possible for $b = (4, 6, c)$ if $c =$ ______.

20. Normally 4 "planes" in four-dimensional space meet at a ______. Normally 4 column vectors in four-dimensional space can combine to produce b. What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $b = (3, 3, 3, 2)$? What 4 equations for x, y, z, t are you solving?

21. When equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution?

22. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, *show that* (a, c) *is a multiple of* (b, d). This is surprisingly important: call it a challenge question. You could use numbers first to see how a, b, c, and d are related. The question will lead to:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has dependent rows then it has dependent columns.}$$

23. In these equations, the third column (multiplying w) is the *same* as the right side b. The column form of the equations *immediately* gives what solution for (u, v, w)?

$$\begin{aligned} 6u + 7v + 8w &= 8 \\ 4u + 5v + 9w &= 9 \\ 2u - 2v + 7w &= 7. \end{aligned}$$

1.3 AN EXAMPLE OF GAUSSIAN ELIMINATION

The way to understand elimination is by example. We begin in three dimensions:

$$\textbf{Original system} \qquad \begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \phantom{{}+2w} &= -2 \\ -2u + 7v + 2w &= 9. \end{aligned} \tag{1}$$

The problem is to find the unknown values of u, v, and w, and we shall apply Gaussian elimination. (Gauss is recognized as the greatest of all mathematicians, but certainly not because of this invention, which probably took him ten minutes. Ironically, it is the most frequently used of all the ideas that bear his name.) The method starts by ***subtracting multiples of the first equation from the other equations***. The goal is to *eliminate u from the last two equations*. This requires that we

(a) subtract 2 times the first equation from the second
(b) subtract -1 times the first equation from the third.

$$\textbf{Equivalent system} \qquad \begin{aligned} \mathbf{2}u + v + w &= 5 \\ -\mathbf{8}v - 2w &= -12 \\ 8v + 3w &= 14. \end{aligned} \tag{2}$$

The coefficient 2 is the ***first pivot***. Elimination is constantly dividing the pivot into the numbers underneath it, to find out the right multipliers.

The pivot for **the second stage of elimination** is -8. We now ignore the first equation. A multiple of the second equation will be subtracted from the remaining equations (in this case there is only the third one) so as to eliminate v. We add the second equation to the third or, in other words, we

(c) subtract -1 times the second equation from the third.

The elimination process is now complete, at least in the "forward" direction:

$$\textbf{Triangular system} \qquad \begin{aligned} \mathbf{2}u + v + w &= 5 \\ -\mathbf{8}v - 2w &= -12 \\ \mathbf{1}w &= 2. \end{aligned} \tag{3}$$

This system is solved backward, bottom to top. The last equation gives $w = 2$. Substituting into the second equation, we find $v = 1$. Then the first equation gives $u = 1$. This process is called ***back-substitution***.

To repeat: Forward elimination produced the pivots $2, -8, 1$. It subtracted multiples of each row from the rows beneath. It reached the "triangular" system (3), which is solved in reverse order: Substitute each newly computed value into the equations that are waiting.

Remark One good way to write down the forward elimination steps is to include the right-hand side as an extra column. There is no need to copy u and v and w and $=$ at every step, so we are left with the bare minimum:

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

At the end is the triangular system, ready for back-substitution. You may prefer this arrangement, which guarantees that operations on the left-hand side of the equations are also done on the right-hand side—because *both sides are there together.*

In a larger problem, forward elimination takes most of the effort. We use multiples of the first equation to produce zeros below the first pivot. Then the second column is cleared out below the second pivot. The forward step is finished when the system is triangular; equation n contains only the last unknown multiplied by the last pivot.

Back-substitution yields the complete solution in the opposite order—beginning with the last unknown, then solving for the next to last, and eventually for the first.

By definition, ***pivots cannot be zero***. We need to divide by them.

The Breakdown of Elimination

Under what circumstances could the process break down? Something *must* go wrong in the singular case, and something *might* go wrong in the nonsingular case. This may seem a little premature—after all, we have barely got the algorithm working. But the possibility of breakdown sheds light on the method itself.

The answer is: With a full set of n pivots, there is only one solution. The system is nonsingular, and it is solved by forward elimination and back-substitution. But ***if a zero appears*** in a pivot position, elimination has to stop—either temporarily or permanently. The system might or might not be singular.

If the first coefficient is zero, in the upper left corner, the elimination of u from the other equations will be impossible. The same is true at every intermediate stage. Notice that a zero can appear in a pivot position, even if the original coefficient in that place was not zero. Roughly speaking, ***we do not know whether a zero will appear until we try***, by actually going through the elimination process.

In many cases this problem can be cured, and elimination can proceed. Such a system still counts as nonsingular; it is only the algorithm that needs repair. In other cases a breakdown is unavoidable. Those incurable systems are singular, they have no solution or else infinitely many, and a full set of pivots cannot be found.

Example 1 Nonsingular (cured by exchanging equations 2 and 3)

$$\begin{aligned} u + v + w &= \_ \\ 2u + 2v + 5w &= \_ \\ 4u + 6v + 8w &= \_ \end{aligned} \quad \longrightarrow \quad \begin{aligned} u + v + w &= \_ \\ 3w &= \_ \\ 2v + 4w &= \_ \end{aligned} \quad \longrightarrow \quad \begin{aligned} u + v + w &= \_ \\ 2v + 4w &= \_ \\ 3w &= \_ \end{aligned}$$

The system is now triangular, and back-substitution will solve it.

Example 2 Singular (incurable)

$$\begin{aligned} u + v + w &= \_ \\ 2u + 2v + 5w &= \_ \\ 4u + 4v + 8w &= \_ \end{aligned} \quad \longrightarrow \quad \begin{aligned} u + v + w &= \_ \\ 3w &= \_ \\ 4w &= \_ \end{aligned}$$

There is no exchange of equations that can avoid zero in the second pivot position. The equations themselves may be solvable or unsolvable. If the last two equations are $3w = 6$ and $4w = 7$, there is no solution. If those two equations happen to be consistent—as in $3w = 6$ and $4w = 8$—then this singular case has an infinity of solutions. We know that $w = 2$, but the first equation cannot decide both u and v.

Section 1.5 will discuss row exchanges when the system is not singular. Then the exchanges produce a full set of pivots. Chapter 2 admits the singular case, and limps forward with elimination. The $3w$ can still eliminate the $4w$, and we will call 3 the second pivot. (There won't be a third pivot.) For the present we trust all n pivot entries to be nonzero, without changing the order of the equations. That is the best case, with which we continue.

The Cost of Elimination

Our other question is very practical. *How many separate arithmetical operations does elimination require, for n equations in n unknowns?* If n is large, a computer is going to take our place in carrying out the elimination. Since all the steps are known, we should be able to predict the number of operations.

For the moment, ignore the right-hand sides of the equations, and count only the operations on the left. These operations are of two kinds. We divide by the pivot to find out what multiple (say ℓ) of the pivot equation is to be subtracted. When we do this subtraction, we continually meet a "multiply–subtract" combination; the terms in the pivot equation are multiplied by ℓ, and then subtracted from another equation.

Suppose we call each division, and each multiplication–subtraction, one operation. In column 1, *it takes n operations for every zero we achieve*—one to find the multiple ℓ, and the other to find the new entries along the row. There are $n-1$ rows underneath the first one, so the first stage of elimination needs $n(n-1) = n^2 - n$ operations. (Another approach to $n^2 - n$ is this: *All n^2 entries need to be changed, except the n in the first row*.) Later stages are faster because the equations are shorter.

When the elimination is down to k equations, only $k^2 - k$ operations are needed to clear out the column below the pivot—by the same reasoning that applied to the first stage, when k equaled n. Altogether, the total number of operations is the sum of $k^2 - k$ over all values of k from 1 to n:

$$\textbf{Left side} \qquad (1^2 + \cdots + n^2) - (1 + \cdots + n) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n^3 - n}{3}.$$

Those are standard formulas for the sums of the first n numbers and the first n squares. Substituting $n = 1$ and $n = 2$ and $n = 100$ into the formula $\frac{1}{3}(n^3 - n)$, forward elimination can take no steps or two steps or about a third of a million steps:

> If n is at all large, ***a good estimate for the number of operations is*** $\frac{1}{3}n^3$.

If the size is doubled, and few of the coefficients are zero, the cost is multiplied by 8.

Back-substitution is considerably faster. The last unknown is found in only one operation (a division by the last pivot). The second to last unknown requires two operations, and so on. Then the total for back-substitution is $1 + 2 + \cdots + n$.

Forward elimination also acts on the right-hand side (subtracting the same multiples as on the left to maintain correct equations). This starts with $n - 1$ subtractions of the first equation. Altogether ***the right-hand side is responsible for n^2 operations***—much less than the $n^3/3$ on the left. The total for forward and back is

$$\textbf{Right side} \qquad [(n-1) + (n-2) + \cdots + 1] + [1 + 2 + \cdots + n] = n^2.$$

Thirty years ago, almost every mathematician would have guessed that a general system of order n could not be solved with much fewer than $n^3/3$ multiplications. (There were even theorems to demonstrate it, but they did not allow for all possible methods.) Astonishingly, that guess has been proved wrong. *There now exists a method that requires only $Cn^{\log_2 7}$ multiplications*! It depends on a simple fact: Two combinations of two vectors in two-dimensional space would seem to take 8 multiplications, but they can be done in 7. That lowered the exponent from $\log_2 8$, which is 3, to $\log_2 7 \approx 2.8$. This discovery produced tremendous activity to find the smallest possible power of n. The exponent finally fell (at IBM) below 2.376. Fortunately for elimination, the constant C is so large and the coding is so awkward that the new method is largely (or entirely) of theoretical interest. The newest problem is the cost with *many processors in parallel.*

Problem Set 1.3

Problems 1–9 are about elimination on 2 by 2 systems.

1. What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11. \end{aligned}$$

After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 and 11 have no influence on those pivots.

√**2.** Solve the triangular system of Problem 1 by back-substitution, y before x. Verify that x times $(2, 10)$ plus y times $(3, 9)$ equals $(1, 11)$. If the right-hand side changes to $(4, 44)$, what is the new solution?

3. What multiple of equation 2 should be *subtracted* from equation 3?

$$\begin{aligned} 2x - 4y &= 6 \\ -x + 5y &= 0. \end{aligned}$$

After this elimination step, solve the triangular system. If the right-hand side changes to $(-6, 0)$, what is the new solution?

√**4.** What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} ax + by &= f \\ cx + dy &= g. \end{aligned}$$

The first pivot is a (assumed nonzero). Elimination produces what formula for the second pivot? What is y? The second pivot is missing when $ad = bc$.

5. Choose a right-hand side which gives no solution and another right-hand side which gives infinitely many solutions. What are two of those solutions?

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \underline{\quad}. \end{aligned}$$

6. Choose a coefficient b that makes this system singular. Then choose a right-hand side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

7. For which numbers a does elimination break down (a) permanently, and (b) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6. \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row exchange.

8. For which three numbers k does elimination break down? Which is fixed by a row exchange? In each case, is the number of solutions 0 or 1 or ∞?

$$\begin{aligned} kx + 3y &= 6 \\ 3x + ky &= -6. \end{aligned}$$

9. What test on b_1 and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

Problems 10–19 study elimination on 3 by 3 systems (and possible failure).

10. Reduce this system to upper triangular form by two row operations:

$$\begin{aligned} 2x + 3y + z &= 8 \\ 4x + 7y + 5z &= 20 \\ -2y + 2z &= 0. \end{aligned}$$

Circle the pivots. Solve by back-substitution for z, y, x.

11. Apply elimination (circle the pivots) and back-substitution to solve

$$\begin{aligned} 2x - 3y \phantom{{}+z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract ___ times row ___ from row ___.

12. Which number d forces a row exchange, and what is the triangular system (not singular) for that d? Which d makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

13. Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a nonzero solution x, y, z.

$$\begin{aligned} x + by \phantom{{}-z} &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0. \end{aligned}$$

14. (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.

(b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.

15. If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

$$\begin{aligned} 2x - y + z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + z &= 2 \end{aligned} \qquad\qquad \begin{aligned} 2x + 2y + z &= 0 \\ 4x + 4y + z &= 0 \\ 6x + 6y + z &= 2. \end{aligned}$$

16. Construct a 3 by 3 example that has 9 different coefficients on the left-hand side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $b = (1, 10, 100)$ and how many with $b = (0, 0, 0)$?

17. Which number q makes this system singular and which right-hand side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

18. (Recommended) It is impossible for a system of linear equations to have exactly two solutions. *Explain why.*

(a) If (x, y, z) and (X, Y, Z) are two solutions, what is another one?

(b) If 25 planes meet at two points, where else do they meet?

19. Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of A is a ______ of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

Problems 20–22 move up to 4 by 4 and *n* by *n*.

20. Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x + y &= 0 \\ x + 2y + z &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5. \end{aligned}$$

21. If you extend Problem 20 following the 1, 2, 1 pattern or the $-1, 2, -1$ pattern, what is the fifth pivot? What is the nth pivot?

22. Apply elimination and back-substitution to solve

$$\begin{aligned} 2u + 3v &= 0 \\ 4u + 5v + w &= 3 \\ 2u - v - 3w &= 5. \end{aligned}$$

What are the pivots? List the three operations in which a multiple of one row is subtracted from another.

23. For the system

$$\begin{aligned} u + v + w &= 2 \\ u + 3v + 3w &= 0 \\ u + 3v + 5w &= 2, \end{aligned}$$

what is the triangular system after forward elimination, and what is the solution?

24. Solve the system and find the pivots when

$$\begin{aligned} 2u - v \qquad\qquad &= 0 \\ -u + 2v - w \qquad &= 0 \\ - v + 2w - z &= 0 \\ - w + 2z &= 5. \end{aligned}$$

You may carry the right-hand side as a fifth column (and omit writing u, v, w, z until the solution at the end).

25. Apply elimination to the system

$$\begin{aligned} u + v + w &= -2 \\ 3u + 3v - w &= 6 \\ u - v + w &= -1. \end{aligned}$$

When a zero arises in the pivot position, exchange that equation for the one below it and proceed. What coefficient of v in the third equation, in place of the present -1, would make it impossible to proceed—and force elimination to break down?

26. Solve by elimination the system of two equations

$$\begin{aligned} x - y &= 0 \\ 3x + 6y &= 18. \end{aligned}$$

Draw a graph representing each equation as a straight line in the x-y plane; the lines intersect at the solution. Also, add one more line—the graph of the new second equation which arises after elimination.

27. Find three values of a for which elimination breaks down, temporarily or permanently, in

$$\begin{aligned} au + v &= 1 \\ 4u + av &= 2. \end{aligned}$$

Breakdown at the first step can be fixed by exchanging rows—but not breakdown at the last step.

28. True or false:

(a) If the third equation starts with a zero coefficient (it begins with $0u$) then no multiple of equation 1 will be subtracted from equation 3.

(b) If the third equation has zero as its second coefficient (it contains $0v$) then no multiple of equation 2 will be subtracted from equation 3.

(c) If the third equation contains $0u$ *and* $0v$, then no multiple of equation 1 or equation 2 will be subtracted from equation 3.

29. (Very optional) Normally the multiplication of two complex numbers

$$(a+ib)(c+id) = (ac-bd) + i(bc+ad)$$

involves the four separate multiplications ac, bd, bc, ad. Ignoring i, can you compute $ac-bd$ and $bc+ad$ with only three multiplications? (You may do additions, such as forming $a+b$ before multiplying, without any penalty.)

30. Use elimination to solve

$$\begin{aligned} u + v + w &= 6 \\ u + 2v + 2w &= 11 \\ 2u + 3v - 4w &= 3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} u + v + w &= 7 \\ u + 2v + 2w &= 10 \\ 2u + 3v - 4w &= 3. \end{aligned}$$

31. For which three numbers a will elimination fail to give three pivots?

$$\begin{aligned} ax + 2y + 3z &= b_1 \\ ax + ay + 4z &= b_2 \\ ax + ay + az &= b_3. \end{aligned}$$

32. Find experimentally the average size (absolute value) of the first and second and third pivots for MATLAB's lu(rand(3, 3)). The average of the first pivot from abs(A(1, 1)) should be 0.5.

1.4 MATRIX NOTATION AND MATRIX MULTIPLICATION

With our 3 by 3 example, we are able to write out all the equations in full. We can list the elimination steps, which subtract a multiple of one equation from another and reach a triangular matrix. For a large system, this way of keeping track of elimination would be hopeless; a much more concise record is needed.

We now introduce **matrix notation** to describe the original system, and **matrix multiplication** to describe the operations that make it simpler. Notice that three different types of quantities appear in our example:

$$\begin{array}{ll} \textbf{Nine coefficients} & 2u + v + w = 5 \\ \textbf{Three unknowns} & 4u - 6v \qquad = -2 \\ \textbf{Three right-hand sides} & -2u + 7v + 2w = 9 \end{array} \qquad (1)$$

On the right-hand side is the column vector b. On the left-hand side are the unknowns u, v, w. Also on the left-hand side are nine coefficients (one of which happens to be zero). It is natural to represent the three unknowns by a vector:

$$\text{The unknown is } x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \text{The solution is } x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

The nine coefficients fall into three rows and three columns, producing a **3 *by* 3 *matrix***:

$$\textbf{Coefficient matrix} \qquad A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}.$$

A is a *square* matrix, because the number of equations equals the number of unknowns. If there are n equations in n unknowns, we have a square n by n matrix. More generally, we might have m equations and n unknowns. Then A is *rectangular*, with m rows and n columns. It will be an "m by n matrix."

Matrices are added to each other, or multiplied by numerical constants, exactly as vectors are—one entry at a time. In fact we may regard vectors as special cases of matrices; *they are matrices with only one column.* As with vectors, two matrices can be added only if they have the same shape:

Addition $A + B$
Multiplication $2A$
$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 1 \\ 1 & 6 \end{bmatrix} \qquad 2\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & 0 \\ 0 & 8 \end{bmatrix}.$$

Multiplication of a Matrix and a Vector

We want to rewrite the three equations with three unknowns u, v, w in the simplified matrix form $Ax = b$. Written out in full, matrix times vector equals vector:

Matrix form $Ax = b$
$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}. \tag{2}$$

The right-hand side b is the column vector of "inhomogeneous terms." **The left-hand side is A times x.** This multiplication will be defined *exactly so as to reproduce the original system*. The first component of Ax comes from "multiplying" the first row of A into the column vector x:

Row times column
$$\begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2u + v + w \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}. \tag{3}$$

The second component of the product Ax is $4u - 6v + 0w$, from the second row of A. The matrix equation $Ax = b$ is equivalent to the three simultaneous equations in equation (1).

Row times column is fundamental to all matrix multiplications. From two vectors it produces a single number. This number is called the ***inner product*** of the two vectors. In other words, the product of a 1 by n matrix (a *row vector*) and an n by 1 matrix (a *column vector*) is a 1 by 1 matrix:

Inner product
$$\begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 5 \end{bmatrix}.$$

This confirms that the proposed solution $x = (1, 1, 2)$ does satisfy the first equation.

There are two ways to multiply a matrix A and a vector x. One way is *a row at a time*. Each row of A combines with x to give a component of Ax. There are three inner products when A has three rows:

***Ax* by rows**
$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 1 \cdot 5 + 6 \cdot 0 \\ 3 \cdot 2 + 0 \cdot 5 + 3 \cdot 0 \\ 1 \cdot 2 + 1 \cdot 5 + 4 \cdot 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}. \tag{4}$$

That is how Ax is usually explained, but the second way is equally important. In fact it is more important! It does the multiplication *a column at a time*. The product Ax is found all at once, as ***a combination of the three columns of A***:

$$\textbf{Ax by columns} \qquad 2\begin{bmatrix}1\\3\\1\end{bmatrix}+5\begin{bmatrix}1\\0\\1\end{bmatrix}+0\begin{bmatrix}6\\3\\4\end{bmatrix}=\begin{bmatrix}7\\6\\7\end{bmatrix}. \tag{5}$$

The answer is twice column 1 plus 5 times column 2. It corresponds to the "column picture" of linear equations. If the right-hand side b has components 7, 6, 7, then the solution has components 2, 5, 0. Of course the row picture agrees with that (and we eventually have to do the same multiplications).

The column rule will be used over and over, and we repeat it for emphasis:

1A Every product Ax can be found using whole columns as in equation (5). Therefore Ax is ***a combination of the columns of A***. The coefficients are the components of x.

Arv

To multiply A times x in n dimensions, we need a notation for the individual entries in A. *The entry in the ith row and jth column is always denoted by a_{ij}.* The first subscript gives the row number, and the second subscript indicates the column. (In equation (4), a_{21} is 3 and a_{13} is 6.) If A is an m by n matrix, then the index i goes from 1 to m—there are m rows—and the index j goes from 1 to n. Altogether the matrix has mn entries, and a_{mn} is in the lower right corner.

One subscript is enough for a vector. The jth component of x is denoted by x_j. (The multiplication above had $x_1 = 2$, $x_2 = 5$, $x_3 = 0$.) Normally x is written as a column vector—like an n by 1 matrix. But sometimes it is printed on a line, as in $x = (2, 5, 0)$. The parentheses and commas emphasize that it is not a 1 by 3 matrix. It is a column vector, and it is just temporarily lying down.

To describe the product Ax, we use the "*sigma*" symbol Σ for summation:

$$\textbf{Sigma notation} \qquad \text{The } i\text{th component of } Ax \text{ is } \sum_{j=1}^{n} a_{ij}x_j.$$

This sum takes us along the ith row of A. The column index j takes each value from 1 to n and we add up the results—the sum is $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$.

We see again that the length of the rows (the number of columns in A) must match the length of x. **An m by n matrix multiplies an n-dimensional vector** (and produces an m-dimensional vector). Summations are simpler than writing everything out in full, but matrix notation is better. (Einstein used "tensor notation," in which a repeated index automatically means summation. He wrote $a_{ij}x_j$ or even $a_i^j x_j$, without the Σ. Not being Einstein, we keep the Σ.)

The Matrix Form of One Elimination Step

So far we have a convenient shorthand $Ax = b$ for the original system of equations. What about the operations that are carried out during elimination? In our example, the first step subtracted 2 times the first equation from the second. On the right-hand side, 2 times the first component of b was subtracted from the second component. *The same result is achieved if we multiply b by this elementary matrix* (or *elimination matrix*):

$$\textbf{Elementary matrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is verified just by obeying the rule for multiplying a matrix and a vector:

$$Eb = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ \mathbf{-12} \\ 9 \end{bmatrix}.$$

The components 5 and 9 stay the same (because of the 1, 0, 0 and 0, 0, 1 in the rows of E). The new second component -12 appeared after the first elimination step.

It is easy to describe the matrices like E, which carry out the separate elimination steps. We also notice the "identity matrix," which does nothing at all.

1B The ***identity matrix*** $\boldsymbol{I}$, with 1s on the diagonal and 0s everywhere else, leaves every vector unchanged. The ***elementary matrix*** $\boldsymbol{E_{ij}}$ subtracts ℓ times row j from row i. This E_{ij} includes $-\ell$ in row i, column j.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ has } Ib = b \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell & 0 & 1 \end{bmatrix} \text{ has } E_{31}b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - \ell b_1 \end{bmatrix}.$$

$Ib = b$ is the matrix analogue of multiplying by 1. A typical elimination step multiplies by E_{31}. The important question is: What happens to A on the left-hand side?

To maintain equality, we must apply the same operation to both sides of $Ax = b$. In other words, we must also multiply the vector Ax by the matrix E. Our original matrix E subtracts 2 times the first component from the second. After this step the new and simpler system (equivalent to the old) is just $E(Ax) = Eb$. It is simpler because of the zero that was created below the first pivot. It is equivalent because we can recover the original system (by adding 2 times the first equation back to the second). So the two systems have exactly the same solution x.

Matrix Multiplication

Now we come to the most important question: *How do we multiply two matrices*? There is a partial clue from Gaussian elimination: We know the original coefficient matrix A, we know the elimination matrix E, and we know the result EA after the elimination step. We hope and expect that

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ times } A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \text{ gives } EA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}.$$

Twice the first row of A has been subtracted from the second row. Matrix multiplication is consistent with the row operations of elimination. We can write the result either as $E(Ax) = Eb$, applying E to both sides of our equation, or as $(EA)x = Eb$. The matrix EA is constructed exactly so that these equations agree, and we don't need parentheses:

Matrix multiplication $\quad$ (EA times x) equals (E times Ax). We just write EAx.

This is the whole point of an "associative law" like $2 \times (3 \times 4) = (2 \times 3) \times 4$. The law seems so obvious that it is hard to imagine it could be false. But the same could be said of the "commutative law" $2 \times 3 = 3 \times 2$—and for matrices EA is not AE.

There is another requirement on matrix multiplication. We know how to multiply Ax, a matrix and a vector. The new definition should be consistent with that one. When a matrix B contains only a single column x, the matrix–matrix product AB should be identical with the matrix–vector product Ax. *More than that*: When B contains several columns b_1, b_2, b_3, the columns of AB should be Ab_1, Ab_2, Ab_3!

Multiplication by columns $\quad AB = A\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & Ab_3 \end{bmatrix}.$

Our first requirement had to do with rows, and this one is concerned with columns. A third approach is to describe each individual entry in AB and hope for the best. In fact, there is only one possible rule, and I am not sure who discovered it. It makes everything work. It does not allow us to multiply every pair of matrices. If they are square, they must have the same size. If they are rectangular, they must *not* have the same shape; ***the number of columns in A has to equal the number of rows in B***. Then A can be multiplied into each column of B.

If A is m by n, and B is n by p, then multiplication is possible. *The product AB will be m by p*. We now find the entry in row i and column j of AB.

1C The i, j entry of AB is the inner product of the ith row of A and the jth column of B. In Figure 1.7, the 3, 2 entry of AB comes from row 3 and column 2:

$$(AB)_{32} = a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42}. \tag{6}$$

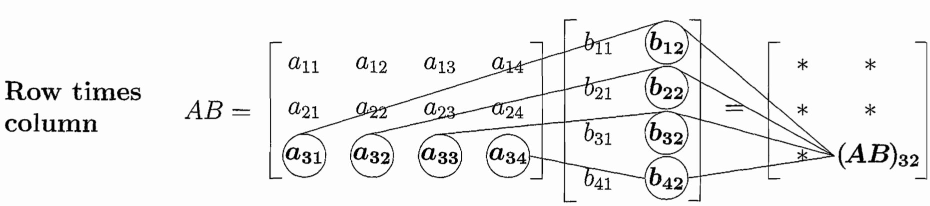


Figure 1.7 A 3 by 4 matrix A times a 4 by 2 matrix B is a 3 by 2 matrix AB.

Note We write AB when the matrices have nothing special to do with elimination. Our earlier example was EA, because of the elementary matrix E. Later we have PA, or LU, or even LDU. The rule for matrix multiplication stays the same.

Example 1

$$AB = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 1 & 0 \\ 4 & 8 & 0 \end{bmatrix}.$$

The entry 17 is $(2)(1) + (3)(5)$, the inner product of the first row of A and first column of B. The entry 8 is $(4)(2) + (0)(-1)$, from the second row and second column.

The third column is zero in B, so it is zero in AB. B consists of three columns side by side, and A multiplies each column separately. ***Every column of AB is a combination of the columns of A.*** Just as in a matrix–vector multiplication, the columns of A are multiplied by the entries in B.

Example 2

Row exchange matrix $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2 & 3 \end{bmatrix}.$$

Example 3 The 1s in the identity matrix I leave every matrix unchanged:

Identity matrix $\quad IA = A \quad$ and $\quad BI = B.$

Important: The multiplication AB can also be done *a row at a time*. In Example 1, the first row of AB uses the numbers 2 and 3 from the first row of A. Those numbers give $2\,[\text{row } 1] + 3\,[\text{row } 2] = [17 \;\; 1 \;\; 0]$. Exactly as in elimination, where all this started, each row of AB is a ***combination of the rows of B***.

We summarize these three different ways to look at matrix multiplication.

1D (i) Each entry of AB is the product of a ***row*** and a ***column***:

$$(AB)_{ij} = (\text{row } i \text{ of } A) \text{ times } (\text{column } j \text{ of } B)$$

(ii) Each column of AB is the product of a ***matrix*** and a ***column***:

$$\text{column } j \text{ of } AB = A \text{ times } (\text{column } j \text{ of } B)$$

(iii) Each row of AB is the product of a ***row*** and a ***matrix***:

$$\text{row } i \text{ of } AB = (\text{row } i \text{ of } A) \text{ times } B.$$

This leads back to a key property of matrix multiplication. Suppose the shapes of three matrices A, B, C (possibly rectangular) permit them to be multiplied. The rows in A and B multiply the columns in B and C. Then the key property is this:

1E Matrix multiplication is associative: $(AB)C = A(BC)$. Just write ABC.

AB times C equals A times BC. If C happens to be just a vector (a matrix with only one column) this is the requirement $(EA)x = E(Ax)$ mentioned earlier. It is the whole basis for the laws of matrix multiplication. And if C has several columns, we have only to think of them placed side by side, and apply the same rule several times. Parentheses are not needed when we multiply several matrices.

There are two more properties to mention—one property that matrix multiplication has, and another which it *does not have*. The property that it does possess is:

1F Matrix operations are distributive:

$$A(B+C)=AB+AC \quad \text{and} \quad (B+C)D=BD+CD.$$

Of course the shapes of these matrices must match properly—B and C have the same shape, so they can be added, and A and D are the right size for premultiplication and postmultiplication. The proof of this law is too boring for words.

The property that fails to hold is a little more interesting:

1G Matrix multiplication is not commutative: Usually $FE \neq EF$.

Example 4 Suppose E subtracts twice the first equation from the second. Suppose F is the matrix for the next step, *to add row* 1 *to row* 3:

$$E=\begin{bmatrix}1&0&0\\ \mathbf{-2}&1&0\\ 0&0&1\end{bmatrix} \quad \text{and} \quad F=\begin{bmatrix}1&0&0\\ 0&1&0\\ \mathbf{1}&0&1\end{bmatrix}.$$

These two matrices do commute and the product does both steps at once:

$$EF=\begin{bmatrix}1&0&0\\ \mathbf{-2}&1&0\\ \mathbf{1}&0&1\end{bmatrix}=FE.$$

In either order, EF or FE, this changes rows 2 and 3 using row 1.

Example 5 Suppose E is the same but G *adds row* 2 *to row* 3. Now the order makes a difference. When we apply E and then G, the second row is altered *before* it affects the third. If E comes *after* G, then the third equation feels no effect from the first. You will see a zero in the (3, 1) entry of EG, where there is a -2 in GE:

$$GE=\begin{bmatrix}1&0&0\\ 0&1&0\\ 0&\mathbf{1}&1\end{bmatrix}\begin{bmatrix}1&0&0\\ \mathbf{-2}&1&0\\ 0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\ -2&1&0\\ \mathbf{-2}&1&1\end{bmatrix} \quad \text{but} \quad EG=\begin{bmatrix}1&0&0\\ -2&1&0\\ \mathbf{0}&1&1\end{bmatrix}.$$

Thus $EG \neq GE$. A random example would show the same thing—most matrices don't commute. Here the matrices have meaning. There was a reason for $EF=FE$, and a reason for $EG \neq GE$. It is worth taking one more step, to see what happens with *all three elimination matrices at once*:

$$GFE=\begin{bmatrix}1&0&0\\ -2&1&0\\ \mathbf{-1}&1&1\end{bmatrix} \quad \text{and} \quad EFG=\begin{bmatrix}1&0&0\\ -2&1&0\\ \mathbf{1}&1&1\end{bmatrix}.$$

The product GFE is the true order of elimination. *It is the matrix that takes the original A to the upper triangular U*. We will see it again in the next section.

The other matrix EFG is nicer. In that order, the numbers -2 from E and 1 from F and G were not disturbed. They went straight into the product. It is the wrong order for elimination. But fortunately *it is the right order for reversing the elimination steps*—which also comes in the next section.

Notice that the product of lower triangular matrices is again lower triangular.

Problem Set 1.4

1. Compute the products

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the third one, draw the column vectors (2, 1) and (0, 3). Multiplying by (1, 1) just adds the vectors (do it graphically).

2. Working a column at a time, compute the products

$$\begin{bmatrix} 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 3 \\ 6 & 6 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}.$$

3. Find two inner products and a matrix product:

$$\begin{bmatrix} 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \end{bmatrix}.$$

The first gives the length of the vector (squared).

4. If an m by n matrix A multiplies an n-dimensional vector x, how many separate multiplications are involved? What if A multiplies an n by p matrix B?

5. Multiply Ax to find a solution vector x to the system $Ax =$ *zero vector*. Can you find more solutions to $Ax = 0$?

$$Ax = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

6. Write down the 2 by 2 matrices A and B that have entries $a_{ij} = i + j$ and $b_{ij} = (-1)^{i+j}$. Multiply them to find AB and BA.

7. Give 3 by 3 examples (not just the zero matrix) of

(a) a diagonal matrix: $a_{ij} = 0$ if $i \neq j$.
(b) a symmetric matrix: $a_{ij} = a_{ji}$ for all i and j.
(c) an upper triangular matrix: $a_{ij} = 0$ if $i > j$.
(d) a skew-symmetric matrix: $a_{ij} = -a_{ji}$ for all i and j.

8. Do these subroutines multiply Ax *by rows or columns*? Start with $B(I) = 0$:

```
   DO 10 I = 1,N                     DO 10 J = 1,N
   DO 10 J = 1,N                     DO 10 I = 1,N
10 B(I) = B(I) + A(I,J) * X(J)    10 B(I) = B(I) + A(I,J) * X(J)
```

The outputs $Bx = Ax$ are the same. The second code is slightly more efficient in FORTRAN and much more efficient on a vector machine (the first changes single entries $B(I)$, the second can update whole vectors).

9. If the entries of A are a_{ij}, use subscript notation to write

(a) the first pivot.
(b) the multiplier ℓ_{i1} of row 1 to be subtracted from row i.
(c) the new entry that replaces a_{ij} after that subtraction.
(d) the second pivot.

10. True or false? Give a specific counterexample when false.

(a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.
(b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.
(c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of AB.
(d) $(AB)^2 = A^2B^2$.

11. The first row of AB is a linear combination of all the rows of B. What are the coefficients in this combination, and what is the first row of AB, if

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}?$$

12. The product of two lower triangular matrices is again lower triangular (all its entries above the main diagonal are zero). Confirm this with a 3 by 3 example, and then explain how it follows from the laws of matrix multiplication.

13. By trial and error find examples of 2 by 2 matrices such that

(a) $A^2 = -I$, A having only real entries.
(b) $B^2 = 0$, although $B \neq 0$.
(c) $CD = -DC$, not allowing the case $CD = 0$.
(d) $EF = 0$, although no entries of E or F are zero.

14. Describe the rows of EA and the *columns* of AE if

$$E = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}.$$

15. Suppose A commutes with every 2 by 2 matrix ($AB = BA$), and in particular

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{commutes with} \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Show that $a = d$ and $b = c = 0$. If $AB = BA$ for all matrices B, then A is a multiple of the identity.

16. Let x be the column vector $(1, 0, \ldots, 0)$. Show that the rule $(AB)x = A(Bx)$ forces the first column of AB to equal A times the first column of B.

17. Which of the following matrices are guaranteed to equal $(A + B)^2$?

$$A^2 + 2AB + B^2, \quad A(A+B) + B(A+B), \quad (A+B)(B+A), \quad A^2 + AB + BA + B^2.$$

18. If A and B are n by n matrices with all entries equal to 1, find $(AB)_{ij}$. Summation notation turns the product AB, and the law $(AB)C = A(BC)$, into

$$(AB)_{ij} = \sum_k a_{ik}b_{kj} \qquad \sum_j \left(\sum_k a_{ik}b_{kj}\right) c_{jl} = \sum_k a_{ik} \left(\sum_j b_{kj}c_{jl}\right).$$

Compute both sides if C is also n by n, with every $c_{jl} = 2$.

19. *A fourth way to multiply matrices is* ***columns of A times rows of B***:

$$AB = (\text{column } 1)(\text{row } 1) + \cdots + (\text{column } n)(\text{row } n) = \text{sum of simple matrices.}$$

Give a 2 by 2 example of this important rule for matrix multiplication.

20. The matrix that rotates the x-y plane by an angle θ is

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Verify that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$ from the identities for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$. What is $A(\theta)$ times $A(-\theta)$?

21. Find the powers A^2, A^3 (A^2 times A), and B^2, B^3, C^2, C^3. What are A^k, B^k, and C^k?

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad C = AB = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Problems 22–31 are about elimination matrices.

22. Write down the 3 by 3 matrices that produce these elimination steps:
(a) E_{21} subtracts 5 times row 1 from row 2.
(b) E_{32} subtracts -7 times row 2 from row 3.
(c) P exchanges rows 1 and 2, then rows 2 and 3.

23. In Problem 22, applying E_{21} and then E_{32} to the column $b = (1, 0, 0)$ gives $E_{32}E_{21}b =$ ___. Applying E_{32} before E_{21} gives $E_{21}E_{32}b =$ ___. When E_{32} comes first, row ___ feels no effect from row ___.

24. Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad E_{32}E_{31}E_{21}A = U.$$

Multiply those E's to get one matrix M that does elimination: $MA = U$.

25. Suppose $a_{33} = 7$ and the third pivot is 5. If you change a_{33} to 11, the third pivot is ___. If you change a_{33} to ___, there is zero in the pivot position.

26. If every column of A is a multiple of $(1, 1, 1)$, then Ax is always a multiple of $(1, 1, 1)$. Do a 3 by 3 example. How many pivots are produced by elimination?

27. What matrix E_{31} subtracts 7 times row 1 from row 3? To reverse that step, R_{31} should ______ 7 times row ___ to row ___. Multiply E_{31} by R_{31}.

28. (a) E_{21} subtracts row 1 from row 2 and then P_{23} exchanges rows 2 and 3. What matrix $M = P_{23}E_{21}$ does both steps at once?

(b) P_{23} exchanges rows 2 and 3 and then E_{31} subtracts row 1 from row 3. What matrix $M = E_{31}P_{23}$ does both steps at once? Explain why the M's are the same but the E's are different.

29. (a) What 3 by 3 matrix E_{13} will add row 3 to row 1?

(b) What matrix adds row 1 to row 3 and *at the same time* adds row 3 to row 1?

(c) What matrix adds row 1 to row 3 and *then* adds row 3 to row 1?

30. Multiply these matrices:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

31. This 4 by 4 matrix needs which elimination matrices E_{21} and E_{32} and E_{43}?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Problems 32–44 are about creating and multiplying matrices.

32. Write these ancient problems in a 2 by 2 matrix form $Ax = b$ and solve them:

(a) X is twice as old as Y and their ages add to 39.

(b) $(x, y) = (2, 5)$ and $(3, 7)$ lie on the line $y = mx + c$. Find m and c.

33. The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$ and $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c).

34. Multiply these matrices in the orders EF and FE and E^2:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}.$$

35. (a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times ______.

(b) Suppose all rows of B are $[1 \;\; 2 \;\; 4]$. Show by example that all rows of EB are *not* $[1 \;\; 2 \;\; 4]$. It is true that those rows are ______.

36. If E adds row 1 to row 2 and F adds row 2 to row 1, does EF equal FE?

37. The first component of Ax is $\sum a_{1j}x_j = a_{11}x_1 + \cdots + a_{1n}x_n$. Write formulas for the third component of Ax and the $(1, 1)$ entry of A^2.

38. If $AB = I$ and $BC = I$, use the associative law to prove $A = C$.

39. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. *All entries are* 1. Which of these matrix operations are allowed, and what are the results?

$$BA \qquad AB \qquad ABD \qquad DBA \qquad A(B + C).$$

40. What rows or columns or matrices do you multiply to find

(a) the third column of AB?
(b) the first row of AB?
(c) the entry in row 3, column 4 of AB?
(d) the entry in row 1, column 1 of CDE?

41. (3 by 3 matrices) Choose the only B so that for every matrix A,

(a) $BA = 4A$.
(b) $BA = 4B$.
(c) BA has rows 1 and 3 of A reversed and row 2 unchanged.
(d) All rows of BA are the same as row 1 of A.

42. True or false?

(a) If A^2 is defined then A is necessarily square.
(b) If AB and BA are defined then A and B are square.
(c) If AB and BA are defined then AB and BA are square.
(d) If $AB = B$ then $A = I$.

43. If A is m by n, how many separate multiplications are involved when

(a) A multiplies a vector x with n components?
(b) A multiplies an n by p matrix B? Then AB is m by p.
(c) A multiplies itself to produce A^2? Here $m = n$.

44. To prove that $(AB)C = A(BC)$, use the column vectors $b_1, \dots, b_n$ of B. First suppose that C has only one column c with entries $c_1, \dots, c_n$:

AB has columns $Ab_1, \dots, Ab_n$ and Bc has one column $c_1b_1 + \cdots + c_nb_n$.

Then $(AB)c = c_1Ab_1 + \cdots + c_nAb_n$ equals $A(c_1b_1 + \cdots + c_nb_n) = A(Bc)$.

Linearity gives equality of those two sums, and $(AB)c = A(Bc)$. The same is true for all other ___ of C. Therefore $(AB)C = A(BC)$.

Problems 45–49 use column–row multiplication and block multiplication.

45. Multiply AB using columns times rows:

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \underline{\qquad\quad} = \underline{\qquad\quad}.$$

46. ***Block multiplication*** separates matrices into blocks (submatrices). If their shapes make block multiplication possible, then it is allowed. Replace these x's by numbers and confirm that block multiplication succeeds.

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} AC + BD \end{bmatrix} \qquad \text{and} \qquad \left[\begin{array}{cc|c} x & x & x \\ x & x & x \\ \hline x & x & x \end{array}\right] \left[\begin{array}{cc|c} x & x & x \\ x & x & x \\ \hline x & x & x \end{array}\right].$$

47. Draw the cuts in A and B and AB to show how each of the four multiplication rules is really a block multiplication to find AB:

(a) Matrix A times columns of B.
(b) Rows of A times matrix B.

(c) Rows of A times columns of B.
(d) Columns of A times rows of B.

48. Block multiplication says that elimination on column 1 produces

$$EA = \begin{bmatrix} 1 & \mathbf{0} \\ -\boldsymbol{c}/a & I \end{bmatrix} \begin{bmatrix} a & \boldsymbol{b} \\ \boldsymbol{c} & D \end{bmatrix} = \begin{bmatrix} a & \boldsymbol{b} \\ \mathbf{0} & \underline{\quad} \end{bmatrix}.$$

49. *Elimination for a 2 by 2 block matrix*: When $A^{-1}A = I$, multiply the first block row by CA^{-1} and subtract from the second row, to find the "*Schur complement*" S:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}.$$

50. With $i^2 = -1$, the product $(A + iB)(x + iy)$ is $Ax + iBx + iAy - By$. Use blocks to separate the real part from the imaginary part that multiplies i:

$$\begin{bmatrix} A & -B \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax - By \\ ? \end{bmatrix} \begin{matrix} \text{real part} \\ \text{imaginary part} \end{matrix}$$

51. Suppose you solve $Ax = b$ for three special right-hand sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the solutions x_1, x_2, x_3 are the columns of a matrix X, what is AX?

52. If the three solutions in Question 51 are $x_1 = (1, 1, 1)$ and $x_2 = (0, 1, 1)$ and $x_3 = (0, 0, 1)$, solve $Ax = b$ when $b = (3, 5, 8)$. Challenge problem: What is A?

53. Find all matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{that satisfy} \quad A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A.$$

54. If you multiply a *northwest matrix* A and a *southeast matrix* B, what type of matrices are AB and BA? "Northwest" and "southeast" mean zeros below and above the antidiagonal going from $(1, n)$ to $(n, 1)$.

55. Write $2x + 3y + z + 5t = 8$ as a matrix A (how many rows?) multiplying the column vector (x, y, z, t) to produce b. The solutions fill a plane in four-dimensional space. *The plane is three-dimensional with no* 4D *volume.*

56. What 2 by 2 matrix P_1 projects the vector (x, y) onto the x axis to produce $(x, 0)$? What matrix P_2 projects onto the y axis to produce $(0, y)$? If you multiply $(5, 7)$ by P_1 and then multiply by P_2, you get (______) and (______).

57. Write the inner product of $(1, 4, 5)$ and (x, y, z) as a matrix multiplication Ax. A has one row. The solutions to $Ax = 0$ lie on a ______ perpendicular to the vector ______. The columns of A are only in ______-dimensional space.

58. In MATLAB notation, write the commands that define the matrix A and the column vectors x and b. What command would test whether or not $Ax = b$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad x = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

59. The MATLAB commands A = eye(3) and v = [3:5]' produce the 3 by 3 identity matrix and the column vector (3, 4, 5). What are the outputs from A * v and v' * v? (Computer not needed!) If you ask for v * A, what happens?

60. If you multiply the 4 by 4 all-ones matrix A = ones(4,4) and the column v = ones(4,1), what is A * v? (Computer not needed.) If you multiply B = eye(4) + ones(4,4) times w = zeros(4,1) + 2 * ones(4,1), what is B * w?

61. Invent a 3 by 3 **magic matrix** M with entries $1, 2, \ldots, 9$. All rows and columns and diagonals add to 15. The first row could be 8, 3, 4. What is M times $(1, 1, 1)$? What is the row vector $[1 \quad 1 \quad 1]$ times M?

1.5 TRIANGULAR FACTORS AND ROW EXCHANGES

We want to look again at elimination, to see what it means in terms of matrices. The starting point was the model system $Ax = b$:

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b. \tag{1}$$

Then there were three elimination steps, with multipliers $2, -1, -1$:

Step 1. Subtract 2 times the first equation from the second;
Step 2. Subtract -1 times the first equation from the third;
Step 3. Subtract -1 times the second equation from the third.

The result was an equivalent system $Ux = c$, with a new coefficient matrix U:

Upper triangular
$$Ux = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} = c. \tag{2}$$

This matrix U is ***upper triangular***—all entries below the diagonal are zero.

The new right side c was derived from the original vector b by the same steps that took A into U. *Forward elimination* amounted to three row operations:

Start with A and b;
Apply steps 1, 2, 3 in that order;
End with U and c.

$Ux = c$ is solved by back-substitution. Here we concentrate on connecting A to U.

The matrices E for step 1, F for step 2, and G for step 3 were introduced in the previous section. They are called ***elementary matrices***, and it is easy to see how they work. To subtract a multiple ℓ of equation j from equation i, *put the number* $-\ell$ *into the* (i, j) *position*. Otherwise keep the identity matrix, with 1s on the diagonal and 0s elsewhere. Then matrix multiplication executes the row operation.

The result of all three steps is $GFEA = U$. Note that E is the first to multiply A, then F, then G. We could multiply GFE together to find the single matrix that takes A to U (and also takes b to c). It is lower triangular (zeros are omitted):

$$\textbf{From } A \textbf{ to } U \qquad GFE = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -1 & 1 & 1 \end{bmatrix}. \tag{3}$$

This is good, but the most important question is exactly the opposite: How would we get from U back to A? ***How can we undo the steps of Gaussian elimination?***

To undo step 1 is not hard. Instead of subtracting, we *add* twice the first row to the second. (Not twice the second row to the first!) The result of doing both the subtraction and the addition is to bring back the identity matrix:

$$\begin{array}{l}\textbf{Inverse of}\\ \textbf{subtraction}\\ \textbf{is addition}\end{array} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

One operation cancels the other. In matrix terms, one matrix is the ***inverse*** of the other. If the elementary matrix E has the number $-\ell$ in the (i, j) position, then its inverse E^{-1} has $+\ell$ in that position. Thus $E^{-1}E = I$, which is equation (4).

We can invert each step of elimination, by using E^{-1} and F^{-1} and G^{-1}. I think it's not bad to see these inverses now, before the next section. The final problem is to undo the whole process at once, and see what matrix takes U back to A.

Since step 3 was last in going from A to U, its matrix G must be the first to be inverted in the reverse direction. Inverses come in the opposite order! The second reverse step is F^{-1} and the last is E^{-1}:

$$\textbf{From } U \textbf{ back to } A \qquad E^{-1}F^{-1}G^{-1}U = A \quad \text{is} \quad LU = A. \tag{5}$$

You can substitute $GFEA$ for U, to see how the inverses knock out the original steps.

Now we recognize the matrix L that takes U back to A. It is called L, because it is *lower triangular*. And it has a special property that can be seen only by multiplying the three inverse matrices in the right order:

$$E^{-1}F^{-1}G^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix} = L. \tag{6}$$

The special thing is that ***the entries below the diagonal are the multipliers*** $\ell = 2, -1$, and -1. When matrices are multiplied, there is usually no direct way to read off the answer. Here the matrices come in just the right order so that their product can be written down immediately. If the computer stores each multiplier ℓ_{ij}—the number that multiplies the pivot row j when it is subtracted from row i, and produces a zero in the i, j position—then these multipliers give a complete record of elimination.

The numbers ℓ_{ij} fit right into the matrix L that takes U back to A.

1H ***Triangular factorization $A = LU$ with no exchanges of rows.*** L is lower triangular, with 1s on the diagonal. The multipliers ℓ_{ij} (taken from elimination) are below the diagonal. U is the upper triangular matrix which appears after forward elimination. The diagonal entries of U are the pivots.

Example 1

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \text{ goes to } U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \text{ with } L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}. \text{ Then } LU = A.$$

Example 2 (which needs a row exchange)

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{cannot be factored into} \quad A = LU.$$

Example 3 (with all pivots and multipliers equal to 1)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU.$$

From A to U there are subtractions of rows. From U to A there are additions of rows.

Example 4 (when U is the identity and L is the same as A)

$$\textbf{Lower triangular case} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}.$$

The elimination steps on this A are easy: (i) E subtracts ℓ_{21} times row 1 from row 2, (ii) F subtracts ℓ_{31} times row 1 from row 3, and (iii) G subtracts ℓ_{32} times row 2 from row 3. The result is the identity matrix $U = I$. The inverses of E, F, and G will bring back A:

$$\boxed{E^{-1} \text{ applied to } F^{-1} \text{ applied to } G^{-1} \text{ applied to } I \text{ produces } A.}$$

$$\begin{bmatrix} 1 & & \\ \ell_{21} & 1 & \\ & & 1 \end{bmatrix} \text{ times } \begin{bmatrix} 1 & & \\ & 1 & \\ \ell_{31} & & 1 \end{bmatrix} \text{ times } \begin{bmatrix} 1 & & \\ & 1 & \\ & \ell_{32} & 1 \end{bmatrix} \text{ equals } \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}.$$

The order is right for the ℓ's to fall into position. This always happens! Note that parentheses in $E^{-1}F^{-1}G^{-1}$ were not necessary because of the associative law.

$A = LU$: The n by n case

The factorization $A = LU$ is so important that we must say more. It used to be missing in linear algebra courses when they concentrated on the abstract side. Or maybe it was thought to be too hard—but you have got it. If the last Example 4 allows any U instead of the particular $U = I$, we can see how the rule works in general. ***The matrix L, applied to U, brings back A***:

$$\boldsymbol{A = LU} \qquad \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} \text{row 1 of } U \\ \text{row 2 of } U \\ \text{row 3 of } U \end{bmatrix} = \text{original } A. \tag{7}$$

The proof is to *apply the steps of elimination*. On the right-hand side they take A to U. On the left-hand side they reduce L to I, as in Example 4. (The first step subtracts ℓ_{21} times $(1, 0, 0)$ from the second row, which removes ℓ_{21}.) Both sides of (7) end up equal to the same matrix U, and the steps to get there are all reversible. Therefore (7) is correct and $A = LU$.

$A = LU$ is so crucial, and so beautiful, that Problem 8 at the end of this section suggests a second approach. We are writing down 3 by 3 matrices, but you can see how the arguments apply to larger matrices. Here we give one more example, and then put $A = LU$ to use.

Example 5 ($A = LU$, with zeros in the empty spaces)

$$A = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}.$$

That shows how a matrix A with three diagonals has factors L and U with two diagonals. This example comes from an important problem in differential equations (Section 1.7). The second difference in A is a backward difference L times a forward difference U.

One Linear System = Two Triangular Systems

There is a serious practical point about $A = LU$. It is more than just a record of elimination steps; L and U are the right matrices to solve $Ax = b$. In fact A could be thrown away! We go from b to c by forward elimination (this uses L) and we go from c to x by back-substitution (that uses U). We can and should do it without A:

$$\textbf{Splitting of } \boldsymbol{Ax = b} \qquad \textit{First} \quad Lc = b \quad \textit{and then} \quad Ux = c. \tag{8}$$

Multiply the second equation by L to give $LUx = Lc$, which is $Ax = b$. Each triangular system is quickly solved. That is exactly what a good elimination code will do:

1. ***Factor*** (from A find its factors L and U).
2. ***Solve*** (from L and U and b find the solution x).

The separation into ***Factor*** and ***Solve*** means that a series of b's can be processed. The ***Solve*** subroutine obeys equation (8): two triangular systems in $n^2/2$ steps each. **The solution for any new right-hand side b can be found in only n^2 operations**. That is far below the $n^3/3$ steps needed to factor A on the left-hand side.

Example 6 This is the previous matrix A with a right-hand side $b = (1, 1, 1, 1)$.

$$\boldsymbol{Ax = b} \qquad \begin{aligned} x_1 - x_2 &= 1 \\ -x_1 + 2x_2 - x_3 &= 1 \\ - x_2 + 2x_3 - x_4 &= 1 \\ - x_3 + 2x_4 &= 1 \end{aligned} \qquad \text{splits into } Lc = b \text{ and } Ux = c.$$

$$\boldsymbol{Lc = b} \qquad \begin{aligned} c_1 &= 1 \\ -c_1 + c_2 &= 1 \\ - c_2 + c_3 &= 1 \\ - c_3 + c_4 &= 1 \end{aligned} \qquad \text{gives } \boldsymbol{c} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{bmatrix}.$$

$$\boldsymbol{Ux = c} \qquad \begin{aligned} x_1 - x_2 &= 1 \\ x_2 - x_3 &= 2 \\ x_3 - x_4 &= 3 \\ x_4 &= 4 \end{aligned} \qquad \text{gives } \boldsymbol{x} = \begin{bmatrix} \mathbf{10} \\ \mathbf{9} \\ \mathbf{7} \\ \mathbf{4} \end{bmatrix}.$$

For these special "tridiagonal matrices," the operation count drops from n^2 to $2n$. You see how $Lc = b$ is solved *forward* (c_1 comes before c_2). This is precisely what happens during forward elimination. Then $Ux = c$ is solved backward (x_4 before x_3).

Remark 1 The LU form is "unsymmetric" on the diagonal: L has 1s where U has the pivots. This is easy to correct. **Divide out of U a diagonal pivot matrix D:**

$$\textbf{Factor out } \boldsymbol{D} \qquad U = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdot \\ & 1 & u_{23}/d_2 & \cdot \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}. \tag{9}$$

In the last example all pivots were $d_i = 1$. In that case $D = I$. But that was very exceptional, and normally LU is different from LDU (also written LDV).

> ***The triangular factorization can be written A = LDU, where L and U have 1s on the diagonal and D is the diagonal matrix of pivots.***

Whenever you see LDU or LDV, it is understood that U or V has 1s on the diagonal—each row was divided by the pivot in D. Then L and U are treated evenly. An example of LU splitting into LDU is

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ & -2 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix} = LDU.$$

That has the 1s on the diagonals of L and U, and the pivots 1 and -2 in D.

Remark 2 We may have given the impression in describing each elimination step, that the calculations must be done in that order. This is wrong. There is *some* freedom, and there is a "Crout algorithm" that arranges the calculations in a slightly different way.

There is no freedom in the final L, D, and U. That is our main point:

1I If $A = L_1 D_1 U_1$ and also $A = L_2 D_2 U_2$, where the L's are lower triangular with unit diagonal, the U's are upper triangular with unit diagonal, and the D's are diagonal matrices with no zeros on the diagonal, then $L_1 = L_2$, $D_1 = D_2$, $U_1 = U_2$. The LDU factorization and the LU factorization are uniquely determined by A.

The proof is a good exercise with inverse matrices in the next section.

Row Exchanges and Permutation Matrices

We now have to face a problem that has so far been avoided: The number we expect to use as a pivot might be zero. This could occur in the middle of a calculation. It will happen at the very beginning if $a_{11} = 0$. A simple example is

$$\textbf{Zero in the pivot position} \qquad \begin{bmatrix} \mathbf{0} & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

The difficulty is clear; no multiple of the first equation will remove the coefficient 3.

The remedy is equally clear. ***Exchange the two equations***, moving the entry 3 up into the pivot. In this example the matrix would become upper triangular:

$$\textbf{Exchange rows} \qquad \begin{aligned} 3u + 4v &= b_2 \\ 2v &= b_1 \end{aligned}$$

To express this in matrix terms, we need the ***permutation matrix*** P that produces the row exchange. It comes from exchanging the rows of I:

$$\textbf{Permutation} \qquad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}.$$

P has the same effect on b, exchanging b_1 and b_2. *The new system is $PAx = Pb$.* The unknowns u and v are *not* reversed in a row exchange.

A permutation matrix P has the same rows as the identity (in some order). There is a single "1" in every row and column. The most common permutation matrix is $P = I$ (it exchanges nothing). The product of two permutation matrices is another permutation— the rows of I get reordered twice.

After $P = I$, the simplest permutations exchange two rows. Other permutations exchange more rows. **There are $n! = (n)(n-1)\cdots(1)$ permutations of size n.** Row 1 has n choices, then row 2 has $n-1$ choices, and finally the last row has only one choice. We can display all 3 by 3 permutations (there are $3! = (3)(2)(1) = 6$ matrices):

$$\boldsymbol{I} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad \boldsymbol{P}_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \quad \boldsymbol{P}_{32}\boldsymbol{P}_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$\boldsymbol{P}_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \quad \boldsymbol{P}_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \quad \boldsymbol{P}_{21}\boldsymbol{P}_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

There will be 24 permutation matrices of order $n = 4$. There are only two permutation matrices of order 2, namely

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

When we know about inverses and transposes (the next section defines A^{-1} and A^{T}), we discover an important fact: ***P^{-1} is always the same as P^{T}.***

A zero in the pivot location raises two possibilities: ***The trouble may be easy to fix, or it may be serious***. This is decided by looking *below the zero*. If there is a nonzero entry lower down in the same column, then a row exchange is carried out. The nonzero entry becomes the needed pivot, and elimination can get going again:

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{bmatrix} \qquad \begin{aligned} d &= 0 \implies \text{no first pivot} \\ a &= 0 \implies \text{no second pivot} \\ c &= 0 \implies \text{no third pivot.} \end{aligned}$$

If $d = 0$, the problem is incurable and this matrix is ***singular***. There is no hope for a unique solution to $Ax = b$. If d is *not* zero, an exchange P_{13} of rows 1 and 3 will move d into the pivot. However the next pivot position also contains a zero. The number a is now below it (the e above it is useless). If a is not zero then another row exchange P_{23} is called for:

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad P_{23}P_{13}A = \begin{bmatrix} \boldsymbol{d} & e & f \\ 0 & \boldsymbol{a} & b \\ 0 & 0 & \boldsymbol{c} \end{bmatrix}$$

One more point: The permutation $P_{23}P_{13}$ will do both row exchanges at once:

$$\textbf{P_{13} acts first} \qquad P_{23}P_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P.$$

If we had known, we could have multiplied A by P in the first place. With the rows in the right order PA, any nonsingular matrix is ready for elimination.

Elimination in a Nutshell: $PA = LU$

The main point is this: If elimination can be completed with the help of row exchanges, then we can imagine that those exchanges are done first (by P). *The matrix PA will not need row exchanges*. In other words, PA allows the standard factorization into L times U. The theory of Gaussian elimination can be summarized in a few lines:

1J In the ***nonsingular*** case, there is a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. Then $Ax = b$ has a *unique solution*:

With the rows reordered in advance, PA can be factored into LU.

In the ***singular*** case, no P can produce a full set of pivots: elimination fails.

In practice, we also consider a row exchange when the original pivot is *near* zero—even if it is not exactly zero. Choosing a larger pivot reduces the roundoff error.

You have to be careful with L. Suppose elimination subtracts row 1 from row 2, creating $\ell_{21} = 1$. Then suppose it exchanges rows 2 and 3. If that exchange is done in advance, the multiplier will change to $\ell_{31} = 1$ in $PA = LU$.

Example 7

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & \mathbf{0} & 2 \\ 0 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U. \tag{10}$$

That row exchange recovers LU—but now $\ell_{31} = 1$ and $\ell_{21} = 2$:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad PA = LU. \tag{11}$$

In MATLAB, A([r k],:) exchanges row k with row r below it (where the kth pivot has been found). We update the matrices L and P the same way. At the start, $P = I$ and sign $= +1$:

```
A([r k], : ) = A([k r], : );
L([r k],1:k–1) = L([k r],1:k–1);
P([r k], : ) = P([k r], : );
sign = –sign
```

The "**sign**" of P tells whether the number of row exchanges is even (sign $= +1$) or odd (sign $= -1$). A row exchange reverses sign. The final value of sign is the **determinant of P** and it does not depend on the order of the row exchanges.

To summarize: A good elimination code saves L and U and P. Those matrices carry the information that originally came in A—and they carry it in a more usable form. $Ax = b$ reduces to two triangular systems. This is the practical equivalent of the calculation we do next—*to find the inverse matrix A^{-1} and the solution $x = A^{-1}b$.*

Problem Set 1.5

1. When is an upper triangular matrix nonsingular (a full set of pivots)?

2. What multiple ℓ_{32} of row 2 of A will elimination subtract from row 3 of A? Use the factored form

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix}.$$

What will be the pivots? Will a row exchange be required?

3. Multiply the matrix $L = E^{-1}F^{-1}G^{-1}$ in equation (6) by GFE in equation (3):

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{times} \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

Multiply also in the opposite order. *Why are the answers what they are?*

4. Apply elimination to produce the factors L and U for

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}.$$

5. Factor A into LU, and write down the upper triangular system $Ux = c$ which appears after elimination, for

$$Ax = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}.$$

6. Find E^2 and E^8 and E^{-1} if

$$E = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}.$$

7. Find the products FGH and HGF if (with upper triangular zeros omitted)

$$F = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

8. (***Second proof of*** $A = LU$) The third row of U comes from the third row of A by subtracting multiples of rows 1 and 2 (*of* U!):

$$\text{row 3 of } U = \text{row 3 of } A - \ell_{31}(\text{row 1 of } U) - \ell_{32}\,(\text{row 2 of } U).$$

(a) Why are rows of U subtracted off and not rows of A? Answer: Because by the time a pivot row is used, ____.

(b) The equation above is the same as

$$\text{row 3 of } A = \ell_{31}(\text{row 1 of } U) + \ell_{32}(\text{row 2 of } U) + 1\,(\text{row 3 of } U).$$

Which rule for matrix multiplication makes this row 3 of L times U?

The other rows of LU agree similarly with the rows of A.

9. (a) Under what conditions is the following product nonsingular?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Solve the system $Ax = b$ starting with $Lc = b$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b.$$

10. (a) Why does it take approximately $n^2/2$ multiplication–subtraction steps to solve each of $Lc = b$ and $Ux = c$?

(b) How many steps does elimination use in solving 10 systems with the same 60 by 60 coefficient matrix A?

11. Solve as two triangular systems, without multiplying LU to find A:

$$LUx = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

12. How could you factor A into a product UL, upper triangular times lower triangular? Would they be the same factors as in $A = LU$?

13. Solve by elimination, exchanging rows when necessary:

$$\begin{aligned} u + 4v + 2w &= -2 \\ -2u - 8v + 3w &= 32 \\ v + w &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} v + w &= 0 \\ u + v \qquad &= 0 \\ u + v + w &= 1. \end{aligned}$$

Which permutation matrices are required?

14. Write down all six of the 3 by 3 permutation matrices, including $P = I$. Identify their inverses, which are also permutation matrices. The inverses satisfy $PP^{-1} = I$ and are on the same list.

15. Find the $PA = LDU$ factorizations (and check them) for

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

16. Find a 4 by 4 permutation matrix that requires three row exchanges to reach the end of elimination (which is $U = I$).

17. The less familiar form $A = LPU$ exchanges rows only at the end:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \rightarrow L^{-1}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} = PU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}.$$

What is L is this case? Comparing with $PA = LU$ in Box 1J, the multipliers now stay in place (ℓ_{21} is 1 and ℓ_{31} is 2 when $A = LPU$).

18. Decide whether the following systems are singular or nonsingular, and whether they have no solution, one solution, or infinitely many solutions:

$$\begin{aligned} v - w &= 2 \\ u - v \qquad &= 2 \\ u \qquad - w &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} v - w &= 0 \\ u - v \qquad &= 0 \\ u \qquad - w &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} v + w &= 1 \\ u + v \qquad &= 1 \\ u \qquad + w &= 1. \end{aligned}$$

19. Which numbers a, b, c lead to row exchanges? Which make the matrix singular?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ a & 8 & 3 \\ 0 & b & 5 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} c & 2 \\ 6 & 4 \end{bmatrix}.$$

Problems 20–31 compute the factorization $A = LU$ (and also $A = LDU$).

20. Forward elimination changes $\begin{bmatrix}1&1\\1&2\end{bmatrix}x = b$ to a triangular $\begin{bmatrix}1&1\\0&1\end{bmatrix}x = c$:

$$\begin{aligned} x + y &= 5\\ x + 2y &= 7\end{aligned} \quad\longrightarrow\quad \begin{aligned} x + y &= 5\\ y &= 2\end{aligned} \qquad \begin{bmatrix}1&1&\mathbf{5}\\1&2&\mathbf{7}\end{bmatrix} \longrightarrow \begin{bmatrix}1&1&\mathbf{5}\\0&1&\mathbf{2}\end{bmatrix}.$$

That step subtracted $\ell_{21} =$ ______ times row 1 from row 2. The reverse step *adds* ℓ_{21} times row 1 to row 2. The matrix for that reverse step is $L =$ ______. Multiply this L times the triangular system $\begin{bmatrix}1&1\\0&1\end{bmatrix}x = \begin{bmatrix}5\\2\end{bmatrix}$ to get ______ $=$ ______. In letters, L multiplies $Ux = c$ to give ______.

21. (Move to 3 by 3) Forward elimination changes $Ax = b$ to a triangular $Ux = c$:

$$\begin{aligned} x + y + z &= 5\\ x + 2y + 3z &= 7\\ x + 3y + 6z &= 11\end{aligned} \qquad \begin{aligned} x + y + z &= 5\\ y + 2z &= 2\\ 2y + 5z &= 6\end{aligned} \qquad \begin{aligned} x + y + z &= 5\\ y + 2z &= 2\\ z &= 2.\end{aligned}$$

The equation $z = 2$ in $Ux = c$ comes from the original $x + 3y + 6z = 11$ in $Ax = b$ by subtracting $\ell_{31} =$ ___ times equation 1 and $\ell_{32} =$ ___ times the *final* equation 2. Reverse that to recover $[1\ \ 3\ \ 6\ \ 11]$ in $[A\ \ b]$ from the final $[1\ \ 1\ \ 1\ \ 5]$ and $[0\ \ 1\ \ 2\ \ 2]$ and $[0\ \ 0\ \ 1\ \ 2]$ in $[U\ \ c]$:

$$\text{Row 3 of } \begin{bmatrix}A & b\end{bmatrix} = (\ell_{31} \text{ Row } 1 + \ell_{32} \text{ Row } 2 + 1 \text{ Row } 3) \text{ of } \begin{bmatrix}U & c\end{bmatrix}.$$

In matrix notation this is multiplication by L. So $A = LU$ and $b = Lc$.

22. What are the 3 by 3 triangular systems $Lc = b$ and $Ux = c$ from Problem 21? Check that $c = (5, 2, 2)$ solves the first one. Which x solves the second one?

23. What two elimination matrices E_{21} and E_{32} put A into upper triangular form $E_{32}E_{21}A = U$? Multiply by E_{32}^{-1} and E_{21}^{-1} to factor A into $LU = E_{21}^{-1}E_{32}^{-1}U$:

$$A = \begin{bmatrix}1&1&1\\2&4&5\\0&4&0\end{bmatrix}.$$

24. What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} to factor A into LU where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U:

$$A = \begin{bmatrix}1&0&1\\2&2&2\\3&4&5\end{bmatrix}.$$

25. When zero appears in a pivot position, $A = LU$ *is not possible*! (We need nonzero pivots d, f, i in U.) Show directly why these are both impossible:

$$\begin{bmatrix}0&1\\2&3\end{bmatrix} = \begin{bmatrix}1&0\\\ell&1\end{bmatrix}\begin{bmatrix}d&e\\0&f\end{bmatrix} \qquad \begin{bmatrix}1&1&0\\1&1&2\\1&2&1\end{bmatrix} = \begin{bmatrix}1&&\\\ell&1&\\m&n&1\end{bmatrix}\begin{bmatrix}d&e&g\\&f&h\\&&i\end{bmatrix}.$$

26. Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ is not possible. Which c produces zero in the third pivot